Hybrid Tableaux for the Difference Modality

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Abstract

We present the first tableau-based decision procedure for basic hybrid logic with the difference modality. The decision procedure is gracefully degrading in that the less expressive constructs don't pay for the computationally expensive difference modality. The procedure can be specialized to reflexive and transitive frames. Key features of our approach are nominal elimination, pattern-based blocking, and expansion control.

Keywords: hybrid logic, modal logic, difference modality, terminating tableaux, decision procedures

1 Introduction

Modal logic with the difference modality $Dp = \lambda x.\, \exists y.\, x{\neq}y \wedge py$ is an expressive language [2,4]. It can express the global modality $Ep = p \dot{\vee} Dp$ and nominals $!p = E(p \dot{\wedge} \dot{\neg}(Dp))$. Gargov and Goranko [18] show that basic modal logic with D is equivalent with respect to modal definability to basic hybrid logic with E (see also [19,21,11,28,1]).

Tableaux for modal logic with D are not well-understood. In a recent handbook chapter on modal proof theory [15], an unsound tableau calculus for basic modal logic with D is given. A sound and complete tableau calculus for basic modal logic with D is given by Balbiani and Demri [2]. Unfortunately, Balbiani and Demri's calculus does not yield a decision procedure as it does not terminate on all inputs.

Recently, several tableau-based decision procedures for hybrid logic with E have been proposed [7,6,24]. The goal behind this work is the design of modular decision procedures that are gracefully degrading if more expressive constructs like E are used. So far, it has been open whether this approach extends to D [6].

This paper presents a tableau-based decision procedure for basic hybrid logic with E and D that is gracefully degrading. Its key features are a pattern-based blocking condition for the $\diamond$-rule and a substitution rule eliminating nominals. Since

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the substitution rule also eliminates equations, straightforward model constructions suffice. We show how our decision procedure can be adapted to reflexive and transitive frames.

Existing tableau-based decision procedures [7,6] for hybrid logic with global modalities (A and E) rely on recording chronological information concerning the construction of tableaux (e.g., prefix order and urfathers). Our procedure seems to be the first tableau-based decision procedure for hybrid logic with global modalities that terminates without recording chronological information.

The calculus of Balbiani and Demri [2] employs a computationally expensive cut rule. To avoid the general inefficiency coming with this rule, we integrate it into the rule for the dual of D. Thus the costs of the cut rule need only be paid if the dual of D is used.

The paper is organized as follows. We start with a representation of hybrid logic in simple type theory. Next, we present the tableau rules. Then we formulate and prove a weak and a strong model existence theorem. The expandedness conditions of the weak model existence theorem yield blocking conditions for the tableau rules yielding a terminating control we call expansion control. We prove termination and obtain a decision procedure. Then we adapt our results to reflexive and transitive frames. We conclude with a discussion of our approach and related work.

2 Hybrid Logic with E and D

We represent modal logic in simple type theory, which gives us an expressive syntax and a solid foundation. The basic idea of the representation goes back to Gallin [16] and can also be found in Gamut [17] (Section 5.8, two-sorted type theory). Since the type-theoretic representation formalizes the semantics of modal logic at the object level, one can prove meta- and object-level theorems of modal logic with a higher-order theorem prover [3].

We start with two base types B and S. The interpretation of B is fixed and consists of two truth values. The interpretation of S is a nonempty set whose elements are called worlds or states. Given two types σ and τ, the *functional type* $\sigma\tau$ is interpreted as the set of all total functions from the interpretation of σ to the interpretation of τ. We write $\sigma_1\sigma_2\sigma_3$ for $\sigma_1(\sigma_2\sigma_3)$.

We employ three kinds of variables: *Nominal variables* x, y, z of type S, *propositional variables* p, q of type SB, and *relational variables* r of type SSB. Nominal variables are called *nominals* for short. We use the logical constants

$$\bot, \top : \mathrm{B} \qquad \doteq\, : \mathrm{SSB}$$
$$\neg : \mathrm{BB} \qquad \exists, \forall : (\mathrm{SB})\mathrm{B}$$
$$\vee, \wedge, \rightarrow : \mathrm{BBB}$$

Terms are defined as usual. We write st for applications, $\lambda x.s$ for abstractions, and $s_1s_2s_3$ for $(s_1s_2)s_3$. We also use infix notation, e.g., $s \wedge t$ for $(\wedge)st$.

Terms of type B are called *formulas*. We employ some common notational conventions: $\exists x.s$ for $\exists(\lambda x.s)$, $\forall x.s$ for $\forall(\lambda x.s)$, and $x \not\doteq y$ for $\neg(x \doteq y)$.

The formulas of modal logic can be either translated to type-theoretic formulas (as in [16,17,22,24]) or directly represented as terms of type SB (as in [8,3]). Here we use the latter approach, which is more elegant since it models modal syntax directly as higher-order syntax. To do so, we need lifted versions of the Boolean connectives, which are defined as follows:

$$\dot\neg px = \neg(px) \qquad \dot\neg : (\mathrm{SB})\mathrm{SB}$$
$$(p \mathbin{\dot\wedge} q)x = px \wedge qx \qquad \dot\wedge : (\mathrm{SB})(\mathrm{SB})\mathrm{SB}$$
$$(p \mathbin{\dot\vee} q)x = px \vee qx \qquad \dot\vee : (\mathrm{SB})(\mathrm{SB})\mathrm{SB}$$

We can now write terms like $p \mathbin{\dot\wedge} \dot\neg q$, which represent modal formulas. Here are the definitions of the remaining *modal constants* we will use:

$$\langle r\rangle px = \exists y.\, rxy \wedge py \qquad \langle \_ \rangle : (\mathrm{SSB})(\mathrm{SB})\mathrm{SB}$$
$$[r]px = \forall y.\, rxy \rightarrow py \qquad [\_] : (\mathrm{SSB})(\mathrm{SB})\mathrm{SB}$$
$$Epx = \exists p \qquad E : (\mathrm{SB})\mathrm{SB}$$
$$Apx = \forall p \qquad A : (\mathrm{SB})\mathrm{SB}$$
$$Dpx = \exists y.\, x \dot{\neq} y \wedge py \qquad D : (\mathrm{SB})\mathrm{SB}$$
$$\bar{D}px = \forall y.\, x \doteq y \vee py \qquad \bar{D} : (\mathrm{SB})\mathrm{SB}$$
$$\dot{x}y = x \doteq y \qquad \dot{\_} : \mathrm{SSB}$$
$$@xpy = px \qquad @ : \mathrm{S}(\mathrm{SB})\mathrm{SB}$$

We call a term $t : \mathrm{SB}$ *modal* if it has the form

$$t \ ::= \ p \mid \dot\neg t \mid t \circ t \mid \mu r t \mid \nu t \mid \dot{x} \mid @xt$$

where $\circ \in \{\dot\wedge, \dot\vee\}$, $\mu \in \{\langle \_ \rangle, [\_]\}$, and $\nu \in \{E, A, D, \bar{D}\}$.

Our type-theoretic presentation of hybrid logic, in particular of nominals as objects of type S, reveals that the essential extension of the basic modal language introduced in basic hybrid logic is the presence of equality at the object level. In this sense, we can see basic hybrid logic as basic modal logic with equality.

A *modal interpretation* $\mathfrak{M}$ is an interpretation of simple type theory that interprets B as the set $\{0, 1\}$, $\perp$ as 0 (i.e., false), $\top$ as 1 (i.e., true), maps S to a non-empty set, gives the logical constants $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$, $\doteq$ their usual meaning, and satisfies the equations defining the modal constants $\dot\neg$, $\dot\wedge$, $\dot\vee$, $\langle \_ \rangle$, $[\_]$, E, A, D, $\bar{D}$, $\dot{\_}$, and $@$. Instead of $\mathfrak{M}t = 1$ we also write $\mathfrak{M} \vDash t$ and say that $\mathfrak{M}$ *satisfies* t, or that t is *valid* in $\mathfrak{M}$. A formula is called *satisfiable* if it has a satisfying modal interpretation.

We use $\mathcal{H}(D)$ as name for the logic given by modal terms and modal interpretations.

We now give some additional syntactic definitions that are needed for the rest of the paper. A modal term $s : \mathrm{SB}$ is called *normal* if it is in negation normal form, that is, has the form

$$s \ ::= \ p \mid \dot\neg p \mid s \circ s \mid \mu r s \mid \nu s \mid \dot{x} \mid \dot\neg \dot{x} \mid @xs$$

where $\circ \in \{\dot\wedge, \dot\vee\}$, $\mu \in \{\langle \_ \rangle, [\_]\}$ and $\nu \in \{E, A, D, \bar{D}\}$. A formula s is called *normal*

$$\mathcal{R}_{\dot\wedge}\ \frac{(s \mathbin{\dot\wedge} t)x}{sx,\ tx} \qquad \mathcal{R}_{\dot\vee}\ \frac{(s \mathbin{\dot\vee} t)x}{sx \ \mid\ tx} \qquad \mathcal{R}_{\Diamond}\ \frac{\langle r\rangle tx}{rxy,\ ty}\ y \notin \mathcal{V}\Gamma \qquad \mathcal{R}_{\Box}\ \frac{[r]tx \qquad rxy}{ty}$$

$$\mathcal{R}_{E}\ \frac{Etx}{ty}\ y \notin \mathcal{V}\Gamma \qquad \mathcal{R}_{A}\ \frac{Atx}{ty}\ y \in \mathcal{V}\Gamma \qquad \mathcal{R}_{D}\ \frac{Dtx}{x \dot{\neq} y,\ ty}\ y \notin \mathcal{V}\Gamma \qquad \mathcal{R}_{\bar D}\ \frac{\bar Dtx}{x \doteq y \ \mid\ ty}\ y \in \mathcal{V}\Gamma$$

$$\mathcal{R}_{\doteq}\ \frac{x \doteq y \in \Gamma}{\Gamma^x_y} \qquad \mathcal{R}_{N}\ \frac{\dot xy}{x \doteq y} \qquad \mathcal{R}_{\bar N}\ \frac{\dot\neg \dot xy}{x \dot{\neq} y} \qquad \mathcal{R}_{@}\ \frac{@ytx}{ty}$$

Γ is the tableau branch to which a rule is applied.

Fig. 1. Tableau Rules $\mathfrak{C}$

if it has the form

$$s \ ::= \ x \doteq y \mid x \dot{\neq} y \mid rxy \mid tx$$

where t is a normal modal term. A formula of the form rxy is called an *accessibility formula*.

Given a term t, we write $\mathcal{V}t$ for the set of variables that occur free in t, and $|t|$ for the size of t. When necessary, $\mathcal{V}$ is extended to sets of terms in the natural way. For instance, given a set X of terms, $\mathcal{V}X := \bigcup\{\mathcal{V}t \mid t \in X\}$.

We write t^x_s for the term obtained from t by capture-free replacement of the free occurrences of x in t by s. Like in the case of $\mathcal{V}$, the notation for substitution is extended pointwise to sets of terms.

3 Tableau Rules

A *branch* is a non-empty set Γ of normal formulas. We say that a formula s *is on a branch* Γ if $s \in \Gamma$. A branch is *closed* if it contains a formula $x \dot{\neq} x$ or two complementary formulas s and $\dot\neg s$. A branch is *open* if it is not closed. A modal interpretation *satisfies a branch* Γ if it satisfies every formula $s \in \Gamma$. A branch is *satisfiable* if it has a satisfying interpretation. A branch is *unsatisfiable* if it is not satisfiable.

Proposition 3.1 *Every closed branch is unsatisfiable.*

Tableau rules are applied to branches and yield one or two extended branches. A tableau rule is *sound* if, when applied to a branch, this branch is unsatisfiable if and only if each of the extended branches is unsatisfiable. A set of tableau rules is *complete* if repeated application of the rules can reduce every unsatisfiable branch to a set of closed branches. Our goal is a set of tableau rules that, under a suitable control, is terminating and still complete. Together with the control, such a set of tableau rules yields a decision procedure for the validity or unsatisfiability of the modal terms of $\mathcal{H}(D)$. To decide the validity of a modal term s, one computes the negation-normal form t of $\dot\neg s$, selects a nominal $x \notin \mathcal{V}t$, and then applies the tableau rules to the branch $\{tx\}$.

Our decision procedure will employ the set $\mathfrak{C}$ of tableau rules shown in Figure 1. The soundness of the rules is easy to verify.

Proposition 3.2 *The tableau rules $\mathfrak{C}$ are sound.*

If you are familiar with prefixed tableau systems for basic model logic, the rules $\mathcal{R}_{\dot\wedge}$, $\mathcal{R}_{\dot\vee}$, $\mathcal{R}_{\diamond}$, and $\mathcal{R}_{\Box}$ will look familiar. Note that our use of type theoretic syntax yields prefixes and accessibility formulas for free. The rules for the remaining modal constants are derived from the defining equations of the constants in Section 2. There is one trick: We have written the defining equation for $\bar{D}$ in Section 2 with a disjunction rather than an implication so that it induces the right rule.

The rule $\mathcal{R}_{\dot=}$ is a substitution rule that when applied to a nontrivial equation $x \dot= y$ eliminates the nominal x from the branch. Hence we call it *nominal elimination*. In contrast to the other rules, which add formulas, nominal elimination modifies formulas on the branch by replacing all occurrences of a nominal x with occurrences of a nominal y. The use of nominal elimination is crucial for our approach to termination. It also provides for straightforward model existence theorems.

4 Weak Model Existence

To prove completeness of the tableau rules $\mathfrak{C}$ we need a model existence theorem. We start with a naive model existence theorem directly induced by the tableau rules (with the exception of $\mathcal{R}_D$, see the discussion in Section 8). We then refine it in the next section to a strong model existence theorem yielding the completeness of the rules under a terminating control.

We call a normal formula s *expanded* on a branch Γ if one of the following *expandedness conditions* holds:

($\mathcal{E}_{\wedge}$) $s = (t_1 \dot\wedge t_2)x$ and $t_1x, t_2x \in \Gamma$

($\mathcal{E}_{\vee}$) $s = (t_1 \dot\vee t_2)x$ and $t_1x \in \Gamma$ or $t_2x \in \Gamma$

($\mathcal{E}^0_{\diamond}$) $s = \langle r\rangle tx$ and there is some y such that $rxy, ty \in \Gamma$

($\mathcal{E}_{\Box}$) $s = [r]tx$ and for every y such that $rxy \in \Gamma$, $ty \in \Gamma$

($\mathcal{E}_E$) $s = Etx$ and there is some y such that $ty \in \Gamma$

($\mathcal{E}_A$) $s = Atx$ and for every $y \in \mathcal{V}\Gamma$, $ty \in \Gamma$

($\mathcal{E}_D$) $s = Dtx$ and there is some $y \neq x$ such that $ty \in \Gamma$

($\mathcal{E}_{\bar{D}}$) $s = \bar{D}tx$ and for every $y \in \mathcal{V}\Gamma$ either $x \dot= y \in \Gamma$ or $ty \in \Gamma$

($\mathcal{E}_{\dot=}$) $s = x \dot= y$ and $x = y$

($\mathcal{E}_N$) $s = \dot{x}y$ and $x \dot= y \in \Gamma$

($\mathcal{E}_{\bar{N}}$) $s = \dot\neg\dot{x}y$ and $x \dot{\neq} y \in \Gamma$

($\mathcal{E}_{@}$) $s = @ytx$ and $ty \in \Gamma$

A branch Γ is *expanded* if every formula $t \in \Gamma$ is expanded on Γ. A branch Γ is *expanded with respect to* an expandedness condition $\mathcal{E}$ if every formula $t \in \Gamma$ of the form corresponding to $\mathcal{E}$ is expanded on Γ. For instance, Γ is expanded with respect to $\mathcal{E}_{\dot=}$ if and only if every equation on Γ is trivial (i.e., has the form $x \dot= x$).

For every branch Γ we obtain a modal interpretation $\mathfrak{M}^\Gamma$ as follows:

$$
\begin{aligned}
x_0 &= \text{the least variable in } \mathcal{V}\Gamma\\
\mathfrak{M}^\Gamma \mathrm{S} &= \mathcal{V}\Gamma\\
\mathfrak{M}^\Gamma x &= \text{if } x \in \mathcal{V}\Gamma \text{ then } x \text{ else } x_0\\
\mathfrak{M}^\Gamma p &= \lambda x \in \mathcal{V}\Gamma.\ \text{if } px \in \Gamma \text{ then } 1 \text{ else } 0\\
\mathfrak{M}^\Gamma r &= \lambda x \in \mathcal{V}\Gamma.\ \lambda y \in \mathcal{V}\Gamma.\ \text{if } rxy \in \Gamma \text{ then } 1 \text{ else } 0
\end{aligned}
$$

For convenience, we will use relational notation for $\mathfrak{M}^\Gamma r$, that is, treat it as a set of pairs.

Theorem 4.1 (Weak Model Existence) *Let Γ be an open and expanded branch. Then $\mathfrak{M}^\Gamma$ satisfies Γ.*

Proof. Let $t \in \Gamma$. By induction on $|t|$ we prove that $\mathfrak{M}^\Gamma$ satisfies t.

Case $t = px$. Assume $px \in \Gamma$. Then $\mathfrak{M}^\Gamma(px) = \mathfrak{M}^\Gamma px = 1$ by the definition of $\mathfrak{M}^\Gamma p$.

Case $t = \dot\neg px$. Assume $\dot\neg px \in \Gamma$. Since Γ open, $px \notin \Gamma$, i.e., $\mathfrak{M}^\Gamma(px) = \mathfrak{M}^\Gamma px = 0$. Hence $\mathfrak{M}^\Gamma \models \dot\neg px$.

Case $t = x \doteq y$. Assume $x \doteq y \in \Gamma$. By $\mathcal{E}_{\doteq}$, $\mathfrak{M}^\Gamma x = x = y = \mathfrak{M}^\Gamma y$, i.e., $\mathfrak{M}^\Gamma \models x \doteq y$.

Case $t = x \dot\neq y$. Assume $x \dot\neq y \in \Gamma$. Since Γ open, $\mathfrak{M}^\Gamma x = x \neq y = \mathfrak{M}^\Gamma y$, i.e., $\mathfrak{M}^\Gamma \not\models x \doteq y$. Hence $\mathfrak{M}^\Gamma \models x \dot\neq y$.

Case $t = \dot x y$. Assume $\dot x y \in \Gamma$. By $\mathcal{E}_N$, $x \doteq y \in \Gamma$. Since $\mathfrak{M}^\Gamma \models \dot x y \iff \mathfrak{M}^\Gamma \models x \doteq y$, the claim follows by Case $t = x \doteq y$.

Case $t = \dot\neg \dot x y$. Assume $\dot\neg \dot x y \in \Gamma$. By $\mathcal{E}_{\bar N}$, $x \dot\neq y \in \Gamma$. Since $\mathfrak{M}^\Gamma \models \dot\neg \dot x y \iff \mathfrak{M}^\Gamma \models x \dot\neq y$, the claim follows by Case $t = x \dot\neq y$.

Case $t = rxy$. Assume $rxy \in \Gamma$. Then $(x, y) \in \mathfrak{M}^\Gamma r$, i.e., $\mathfrak{M}^\Gamma \models rxy$.

Case $t = @ysx$. Assume $@ysx \in \Gamma$. By $\mathcal{E}_@$, $sy \in \Gamma$. By the inductive hypothesis, $\mathfrak{M}^\Gamma \models sy$. Hence $\mathfrak{M}^\Gamma \models @ysx$.

Case $t = (t_1 \dot\wedge t_2)x$. Assume $(t_1 \dot\wedge t_2)x \in \Gamma$. By $\mathcal{E}_\wedge$, $t_1 x \in \Gamma$ and $t_2 x \in \Gamma$. By the inductive hypothesis, $\mathfrak{M}^\Gamma \models t_1 x$ and $\mathfrak{M}^\Gamma \models t_2 x$, and hence $\mathfrak{M}^\Gamma \models (t_1 \dot\wedge t_2)x$.

Case $t = (t_1 \dot\vee t_2)x$. Analogously to the preceding case.

Case $t = \langle r \rangle sx$. By $\mathcal{E}^0_\Diamond$, there exists some y such that $rxy, sy \in \Gamma$. Hence by induction, $(x, y) \in \mathfrak{M}^\Gamma r$ and $\mathfrak{M}^\Gamma sy = \mathfrak{M}^\Gamma(sy) = 1$. So, y witnesses validity of $\langle r \rangle sx$ in $\mathfrak{M}^\Gamma$.

Case $t = [r]sx$. Assume $[r]sx \in \Gamma$. We have to show that for every pair $(x, y) \in \mathfrak{M}^\Gamma r$ it holds $\mathfrak{M}^\Gamma sy = 1$. So assume $(x, y) \in \mathfrak{M}^\Gamma r$. By the definition of $\mathfrak{M}^\Gamma r$, $rxy \in \Gamma$. Then, by $\mathcal{E}_\Box$, $sy \in \Gamma$. By the inductive hypothesis it holds $\mathfrak{M}^\Gamma sy = \mathfrak{M}^\Gamma(sy) = 1$.

Case $t = Asx$. Assume $Asx \in \Gamma$. To show: $\mathfrak{M}^\Gamma sy = 1$ for all $y \in \mathfrak{M}^\Gamma \mathrm{S}$. So, let $y \in \mathfrak{M}^\Gamma \mathrm{S}$ be arbitrary. By $\mathcal{E}_A$, $sy \in \Gamma$. By the inductive hypothesis, $\mathfrak{M}^\Gamma sy = \mathfrak{M}^\Gamma(sy) = 1$.

Case $t = Esx$. Assume $Esx \in \Gamma$. By $\mathcal{E}_E$, there is some y such that $ty \in \Gamma$. By the inductive hypothesis, $\mathfrak{M}^\Gamma sy = \mathfrak{M}^\Gamma(sy) = 1$, i.e., y witnesses validity of Esx in $\mathfrak{M}^\Gamma$.

Case $t = \bar D sx$. Assume $\bar D sx \in \Gamma$. To show: for every $y \in \mathfrak{M}^\Gamma \mathrm{S}$, either $\mathfrak{M}^\Gamma y = \mathfrak{M}^\Gamma x$

$$A(\langle r\rangle p)x$$
$$\Big|\, \mathcal{R}_A$$
$$\langle r\rangle px$$
$$\Big|\, \mathcal{R}_\Diamond$$
$$rxy,\ py$$
$$\Big|\, \mathcal{R}_A$$
$$\langle r\rangle py$$
$$\Big|\, \mathcal{R}_\Diamond$$
$$\vdots$$

Fig. 2. A Non-terminating Tableau Derivation

or $\mathfrak{M}^\Gamma sy = 1$. So, let $y \in \mathfrak{M}^\Gamma \mathrm{S}$ be arbitrary. By $\mathcal{E}_{\bar{D}}$, either $x \dot{=} y \in \Gamma$ or $ty \in \Gamma$. In the former case, by $\mathcal{E}_{\dot{=}}$ it holds $\mathfrak{M}^\Gamma x = x = y = \mathfrak{M}^\Gamma y$. Otherwise, by the inductive hypothesis, $\mathfrak{M}^\Gamma sy = \mathfrak{M}^\Gamma(sy) = 1$.

Case $t = Dsx$**.** Assume $Dsx \in \Gamma$. By $\mathcal{E}_D$, there is some $y \neq x$ such that $ty \in \Gamma$. Clearly, $\mathfrak{M}^\Gamma x = x \neq y = \mathfrak{M}^\Gamma y$. By the inductive hypothesis, $\mathfrak{M}^\Gamma sy = \mathfrak{M}^\Gamma(sy) = 1$. Consequently, y witnesses validity of Dsx in $\mathfrak{M}^\Gamma$. □

5 Strong Model Existence

Application of the tableau rules stops if either a closed or an open and expanded branch is reached. Hence the expandedness conditions determine whether a terminating control exists. As it turns out, the expandedness condition $\mathcal{E}^0_\Diamond$ coming with the weak model existence theorem does not provide for a terminating control. The problem is caused by the interplay of $\mathcal{R}_A$ and $\mathcal{R}_\Diamond$ and can be seen from the infinite tableau derivation shown in Figure 2. The derivation starts from the satisfiable branch $\{A(\langle r\rangle p)x\}$.

To obtain a terminating control, we will employ an expandedness condition $\mathcal{E}_\Diamond$ for diamond formulas that is weaker than $\mathcal{E}^0_\Diamond$ but still suffices for a model existence theorem. Here is the definition:

($\mathcal{E}_\Diamond$) A formula $\langle r\rangle tx$ is *weakly expanded on a branch* Γ if there are formulas $ryz \in \Gamma$ and $tz \in \Gamma$ such that $[r]sy \in \Gamma$ for all $[r]sx \in \Gamma$.

Note that a diamond formula that is expanded on a branch Γ is also weakly expanded on Γ. To ease our language, we call non-diamond formulas *weakly expanded on* Γ if they are expanded on Γ. A branch Γ is called *weakly expanded* if every formula on Γ is weakly expanded on Γ.

To show that weakly expanded branches are satisfiable if they open, we need the notion of safe accessibility formulas. An accessibility formula rxy is called *safe for a branch* Γ if $sy \in \Gamma$ for all formulas $[r]sx \in \Gamma$.

Proposition 5.1 *Let Γ be an open and weakly expanded branch. If Δ is a set of accessibility formulas that are safe for Γ, then $\Gamma \cup \Delta$ is open and weakly expanded.*

Proof. Adding accessibility formulas rxy does not affect openness. The only expandedness condition that may be affected by adding accessibility formulas is $\mathcal{E}_\Box$. However, since only safe accessibility formulas are added, expandedness with respect to $\mathcal{E}_\Box$ is not destroyed. □

Proposition 5.2 *Let Γ be an open and weakly expanded branch. Then there exists a set Δ of accessibility formulas safe for Γ such that $\Gamma \cup \Delta$ is open and expanded.*

Proof. We choose Δ such that it contains an accessibility formula for every diamond formula on Γ that is not expanded on Γ. Let $\langle r\rangle tx \in \Gamma$ be such a formula. Since $\langle r\rangle tx$ is weakly expanded on Γ, there are formulas $ryz \in \Gamma$ and $tz \in \Gamma$ such that $[r]sy \in \Gamma$ for all $[r]sx \in \Gamma$. Since Γ is expanded with respect to $\mathcal{E}_\Box$, we have $sz \in \Gamma$ for all $[r]sx \in \Gamma$. Hence rxz is safe for Γ. We choose rxz as the accessibility formula for $\langle r\rangle tx \in \Gamma$ on Δ. By Proposition 5.1 we know that $\Gamma \cup \Delta$ is open and weakly expanded. It remains to show that $\langle r\rangle tx$ is expanded on $\Gamma \cup \Delta$. This is the case since $\Gamma \cup \Delta$ contains rxz and tz. □

Theorem 5.3 (Strong Model Existence) *Every open and weakly expanded branch is satisfiable.*

Proof. Follows from Proposition 5.2 and Theorem 4.1 (weak model existence). □

6 Termination

We only consider finite branches in the following. The tableau rules in Figure 1 are applied to a particular formula s on a particular branch Γ. For instance, $\mathcal{R}_D$ is applied to formulas of the form Dtx. We impose the control that a rule can only be applied to a formula s on a branch Γ if Γ is open and s is not weakly expanded on Γ. We refer to this control as *expansion control*. We call a branch Γ *terminal* if no rule can be applied to it under expansion control.

Proposition 6.1 *A terminal branch is either closed or weakly expanded.*

Proof. Straightforward verification of rules weak expandedness conditions. □

We write $\Gamma \to \Delta$ if the branch Δ can be obtained from Γ by a single rule application under expansion control. We have a decision procedure for $\mathcal{H}(D)$ if the relation $\Gamma \to \Delta$ is terminating.

To establish termination, we define a function $\mathcal{C}$ that assigns to every branch a *complexity* $\mathcal{C}\Gamma \in \mathbb{N}^5$ such that $\mathcal{C}\Gamma > \mathcal{C}\Delta$ if $\Gamma \to \Delta$. For $\mathcal{C}\Gamma > \mathcal{C}\Delta$ we choose the terminating lexical order obtained from the terminating order $>$ on the natural numbers $\mathbb{N}$. The component complexities are named as follows:

$$\mathcal{C}\Gamma = (\mathcal{C}_E\Gamma,\ \mathcal{C}_D\Gamma,\ \mathcal{C}_\Diamond\Gamma,\ \mathcal{C}_{\doteq}\Gamma,\ \mathcal{C}_R\Gamma)$$

$\mathcal{C}_E\Gamma$, $\mathcal{C}_D\Gamma$, $\mathcal{C}_\Diamond\Gamma$, and $\mathcal{C}_{\doteq}\Gamma$ measure the contribution of the rules $\mathcal{R}_E$, $\mathcal{R}_D$, $\mathcal{R}_\Diamond$, and $\mathcal{R}_{\doteq}$, respectively. $\mathcal{C}_R\Gamma$ measures the contribution of the remaining rules. An application of $\mathcal{R}_\Diamond$, for instance, will decrease $\mathcal{C}_\Diamond\Gamma$ and not increase $\mathcal{C}_E\Gamma$ and $\mathcal{C}_D\Gamma$.

We write $\mathrm{Mod}\,\Gamma$ for the set of all modal terms occurring on Γ, possibly as subterms. Since Γ is finite, $\mathrm{Mod}\,\Gamma$ is finite. The crucial observation is that all rules but $\mathcal{R}_{\doteq}$ leave $\text{Mod}\,\Gamma$ unchanged, and that $\mathcal{R}_{\doteq}$ does not increase the cardinality of $\text{Mod}\,\Gamma$.

Proposition 6.2 *If* $\Gamma \to \Delta$ *by a rule different from* $\mathcal{R}_{\doteq}$*, then* $\text{Mod}\,\Gamma = \text{Mod}\,\Delta$*. If* $\Gamma \to \Delta$ *by* $\mathcal{R}_{\doteq}$*, then* $|\text{Mod}\,\Gamma| \geq |\text{Mod}\,\Delta|$.

We now define the component complexity

$$\mathcal{C}_E\Gamma \;:=\; |\text{Mod}\,\Gamma - \{\, s \mid \exists x\colon sx \in \Gamma \,\}|$$

Obviously, $\mathcal{C}_E\Gamma$ is decreased by $\mathcal{R}_E$ and not increased by any of the other rules. The definition

$$\begin{aligned}\mathcal{C}_D\Gamma \;:=\;& |\text{Mod}\,\Gamma - \{\, s \mid \exists y\colon sy \in \Gamma \,\}| \\ +\;& |\text{Mod}\,\Gamma - \{\, s \mid \exists x, y\colon \{sx,\, x{\neq}y,\, sy\} \subseteq \Gamma \,\}|\end{aligned}$$

follows the same idea. One can verify that $\mathcal{C}_D\Gamma$ is decreased by $\mathcal{R}_D$ and not increased by any of the other rules. That the second argument of the sum is needed can be seen with the branch $\{Dsx,\, x{\neq}y,\, sy, Dsy\}$ where Dsy is not expanded. To see that $\mathcal{R}_{\doteq}$ does not increase $\mathcal{C}_D\Gamma$ note that Γ is terminal if it contains a disequation $x{\neq}x$.

A *pattern* is a set of the form $\{\langle r\rangle s, [r]t_1, \ldots, [r]t_n\}$ where $n \geq 0$. We use $\text{Pat}\,\Gamma$ to denote the set of all patterns that are subsets of $\text{Mod}\,\Gamma$. $\text{Pat}\,\Gamma$ is a finite set left unchanged by all rules but $\mathcal{R}_{\doteq}$. Moreover, $\mathcal{R}_{\doteq}$ does not increase the cardinality of $\text{Pat}\,\Gamma$. A pattern $\{\langle r\rangle s, [r]t_1, \ldots, [r]t_n\}$ is *expanded* on a branch Γ if there are y, z such that $\{ryz, sz, [r]t_1y, \ldots, [r]t_ny\} \subseteq \Gamma$. If $\mathcal{R}_\diamond$ is applicable under expansion control, it expands a not yet expanded pattern. Hence we define

$$\mathcal{C}_\diamond\Gamma \;:=\; |\text{Pat}\,\Gamma - \{\, P \in \text{Pat}\,\Gamma \mid P \text{ expanded on } \Gamma \,\}|$$

One can verify that $\mathcal{R}_\diamond$ decreases $\mathcal{C}_\diamond\Gamma$ and the other rules do not increase $\mathcal{C}_\diamond\Gamma$.

We have now treated all nominal-introducing rules. The contribution of $\mathcal{R}_{\doteq}$ is that it eliminates a nominal. Hence we define

$$\mathcal{C}_{\doteq}\Gamma \;:=\; \text{number of nominals occurring on } \Gamma$$

Clearly, $\mathcal{R}_{\doteq}$ decreases $\mathcal{C}_{\doteq}\Gamma$ and all other rules but $\mathcal{R}_E$, $\mathcal{R}_D$, and $\mathcal{R}_\diamond$ leave $\mathcal{C}_{\doteq}\Gamma$ unchanged.

The remaining rules leave the nominals and modal subterms of the branch unchanged. They always add a formula $x{\doteq}y$ or $x{\neq}y$ or sx where s is modal. Besides these formulas Γ may also contain formulas rxy. Hence we define

$$\mathcal{C}_R\Gamma \;:=\; 2(\mathcal{C}_{\doteq}\Gamma)^2 + (\text{Rel}\,\Gamma)\cdot(\mathcal{C}_{\doteq}\Gamma)^2 + |\text{Mod}\,\Gamma|\cdot(\mathcal{C}_{\doteq}\Gamma) - |\Gamma|$$

where $\text{Rel}\,\Gamma$ denotes the number of relational variables occurring on Γ. All rules but $\mathcal{R}_E$, $\mathcal{R}_D$, $\mathcal{R}_\diamond$, and $\mathcal{R}_{\doteq}$ decrease $\mathcal{C}_R\Gamma$. This ends our termination proof.

Theorem 6.3 *The tableau rules are terminating under expansion control and yield a decision procedure for* $\mathcal{H}(D)$.

Expansion control imposes the negations of the expandedness conditions as *blocking conditions* for the tableau rules. We refer to the blocking condition imposed by $\mathcal{E}_\Diamond$ as *pattern-based blocking*. We see pattern-based blocking and nominal elimination ($\mathcal{R}_{\dot{=}}$) as the most innovative features of our decision algorithm. Pattern-based blocking is needed to obtain termination in the presence of both A and $\Diamond$.

7 Restricted Frame Classes

Our tableau-based decision procedure can be adapted to frame classes axiomatized by any combination of

4	$\forall r\,\forall p\,\forall x.\ \ (\langle r\rangle(\langle r\rangle p) \dot{\to} \langle r\rangle p)x$	(transitivity)
D	$\forall r\,\forall p\,\forall x.\ \ ([r]p \dot{\to} \langle r\rangle p)x$	(seriality)
T	$\forall r\,\forall p\,\forall x.\ \ (p \dot{\to} \langle r\rangle p)x$	(reflexivity)

where $\dot{\to}$ denotes the lifted version of $\to$, analogously to $\dot{\wedge}$ and $\dot{\vee}$.

Seriality of an accessibility relation r can be enforced by adding the term $A(\langle r\rangle\dot{\top})$ (where $\dot{\top} = \lambda x.\top$) to the branch whose satisfiability is to be decided.

Transitivity and reflexivity require additional rules (cf. [25,15]):

$$\mathcal{R}_4\ \frac{[r]tx \qquad rxy}{[r]ty} \qquad\qquad\qquad \mathcal{R}_{\mathrm{T}}\ \frac{[r]tx}{tx}$$

We define the sets $\mathfrak{C}_{\mathrm{K4}}$, $\mathfrak{C}_{\mathrm{T}}$, and $\mathfrak{C}_{\mathrm{S4}}$ of tableau rules as extensions of $\mathfrak{C}$ (see Figure 1) by, respectively, $\mathcal{R}_4$, $\mathcal{R}_{\mathrm{T}}$, and both $\mathcal{R}_4$ and $\mathcal{R}_{\mathrm{T}}$. For convenience, $\mathfrak{C}$ will also be referred to as $\mathfrak{C}_{\mathrm{K}}$. Note that in the case of $\mathfrak{C}_{\mathrm{S4}}$ we could also remove the rule $\mathcal{R}_\Box$ as it can be simulated by $\mathcal{R}_4$ and $\mathcal{R}_{\mathrm{T}}$.

While the rules $\mathfrak{C}$ are sound with respect to the class K of all frames, the sets $\mathfrak{C}_{\mathrm{K4}}$, $\mathfrak{C}_{\mathrm{T}}$ and $\mathfrak{C}_{\mathrm{S4}}$ are sound with respect to their corresponding frame classes.

Proposition 7.1

(i) *The rules $\mathfrak{C}_{\mathrm{K4}}$ are sound with respect to the frame class* K4 *defined by* **4**.

(ii) *The rules $\mathfrak{C}_{\mathrm{T}}$ are sound with respect to the frame class* T *defined by* **T**.

(iii) *The rules $\mathfrak{C}_{\mathrm{S4}}$ are sound with respect to the frame class* S4 *defined by* **4** *and* **T**.

For each of the new rule sets, the notion of expandedness is adapted by replacing $\mathcal{E}_\Box$ by, respectively:

($\mathcal{E}_\Box^{\mathrm{K4}}$) $[r]tx$ is expanded on Γ if for every y such that $rxy \in \Gamma$, $\{ty, [r]ty\} \subseteq \Gamma$.

($\mathcal{E}_\Box^{\mathrm{T}}$) $[r]tx$ is expanded on Γ if $tx \in \Gamma$ and, for every y such that $rxy \in \Gamma$, $ty \in \Gamma$.

($\mathcal{E}_\Box^{\mathrm{S4}}$) $[r]tx$ is expanded on Γ if $tx \in \Gamma$ and, for every y such that $rxy \in \Gamma$, $[r]ty \in \Gamma$.

It is easy to see that condition $\mathcal{E}_\Box^{\mathrm{S4}}$ is equivalent to the conjunction of $\mathcal{E}_\Box^{\mathrm{K4}}$ and $\mathcal{E}_\Box^{\mathrm{T}}$.

The notions of terminality and (weak) expandedness of branches are adapted to the new rules in the natural way, and written as *$\mathfrak{C}_L$-terminality* and *(weak) $\mathfrak{C}_L$-expandedness*, respectively, for $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$. Expansion control is adapted accordingly.

Termination of the modified calculi is shown in exactly the same way as before, with the new rules treated together with the remaining rules ($\mathcal{C}_R\Gamma$).

As for completeness, all we need is to adapt our definition of safe accessibility formulas according to the individual frame restrictions. So, for $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$, we call a formula rxy *safe* for a branch Γ with respect to $\mathfrak{C}_L$ (*$\mathfrak{C}_L$-safe*) if:

$L = \mathrm{K}$ and $sy \in \Gamma$ for all formulas $[r]sx \in \Gamma$ (same as in Section 5),

$L = \mathrm{K4}$ and $(x, y) \in (\{(x, y) \mid \{sy, [r]sy\} \subseteq \Gamma$ for all formulas $[r]sx \in \Gamma\})^+$,

$L = \mathrm{T}$ and rxy is $\mathfrak{C}_{\mathrm{K}}$-safe for Γ or $x = y$,

$L = \mathrm{S4}$ and $(x, y) \in (\{(x, y) \mid \{sy, [r]sy\} \subseteq \Gamma$ for all formulas $[r]sx \in \Gamma\})^*$.

To deal with transitivity, we need an additional lemma.

Lemma 7.2 *If Γ is weakly $\mathfrak{C}_{\mathrm{K4}}$-expanded, rxy is $\mathfrak{C}_{\mathrm{K4}}$-safe for Γ and $[r]tx \in \Gamma$, then $\{ty, [r]ty\} \subseteq \Gamma$.*

Proof. The formula rxy being $\mathfrak{C}_{\mathrm{K4}}$-safe for Γ means there is some $n \geq 1$ such that $(x, y) \in (\{(z, u) \mid \text{for all } s : [r]sz \in \Gamma \text{ implies } su, [r]su \in \Gamma\})^n$. The claim follows by straightforward induction on n. □

Propositions 5.1 and 5.2 are adapted as follows.

Proposition 7.3 *Let $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$. Let Γ be an open and weakly $\mathfrak{C}_L$-expanded branch. If Δ is a set of accessibility formulas safe for Γ, then $\Gamma \cup \Delta$ is open and weakly $\mathfrak{C}_L$-expanded.*

Proof. The case $L = \mathrm{K}$ is covered by Proposition 5.1, so let us focus on the remaining cases. Adding accessibility formulas does not affect openness of Γ. The only expandedness condition that may be affected by adding accessibility formulas is $\mathcal{E}^L_\Box$. So, assuming $rxy \in \Delta$, it suffices to show that adding rxy to Γ preserves $\mathcal{E}^L_\Box$.

Case $L = \mathrm{K4}$**.** By Lemma 7.2.

Case $L = \mathrm{T}$**.** Then either rxy is $\mathfrak{C}_{\mathrm{K}}$-safe for Γ or $x = y$. In the latter case, it suffices to check that for every formula $[r]tx \in \Gamma$ it holds $tx \in \Gamma$, which is the case by the first part of $\mathcal{E}^{\mathrm{T}}_\Box$. In the former case, we additionally have to ensure that $ty \in \Gamma$, which holds because rxy is $\mathfrak{C}_{\mathrm{K}}$-safe.

Case $L = \mathrm{S4}$**.** Clearly, rxy is $\mathfrak{C}_{\mathrm{S4}}$-safe for Γ if and only if rxy is either $\mathfrak{C}_{\mathrm{T}}$-safe or $\mathfrak{C}_{\mathrm{K4}}$-safe for Γ. Since $\mathcal{E}^{\mathrm{S4}}_\Box$ implies both $\mathcal{E}^{\mathrm{T}}_\Box$ and $\mathcal{E}^{\mathrm{K4}}_\Box$, the claim holds by a straightforward combination of the reasoning used in the preceding two cases. □

Proposition 7.4 *Let $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$. Let Γ be an open and weakly $\mathfrak{C}_L$-expanded branch. If Δ is the set of all accessibility formulas safe for Γ, then $\Gamma \cup \Delta$ is open and $\mathfrak{C}_L$-expanded.*

Proof. By Proposition 7.3, it suffices to show that every formula $\langle r \rangle tx \in \Gamma$ is expanded on $\Gamma \cup \Delta$, which is proven similarly to Proposition 5.2 in all four cases. □

A proof of the following weak model existence theorem can be obtained simply by replacing the notions of expandedness and safe accessibility formulas in the proof of Theorem 4.1 by their parametric versions defined in this section.

Proposition 7.5 *Let $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$. Let Γ be an open and $\mathfrak{C}_L$-expanded branch. Then $\mathfrak{M}_L^\Gamma$ satisfies Γ.*

It remains to show that, for $L \in \{\mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$, the model $\mathfrak{M}_L^\Delta$ of an open and $\mathfrak{C}_L$-expanded branch $\Delta := \Gamma \cup \{rxy \mid rxy\ \mathfrak{C}_L\text{-safe for } \Gamma\}$ belongs to the frame class L. Reflexivity of our model in the cases of $L = \mathrm{T}$ and $L = \mathrm{S4}$ follows from the fact that rxx is $\mathfrak{C}_L$-safe for Γ for every $x \in \mathcal{V}\Gamma$. In the cases of $L = \mathrm{K4}$ and $L = \mathrm{S4}$, we know additionally that the relation $\{(x, y) \mid rxy\ \mathfrak{C}_L\text{-safe for } \Gamma\}$ is transitive. So, to conclude that our entire model is transitive, it suffices to show that a $\mathfrak{C}_L$-expanded branch Γ contains only $\mathfrak{C}_L$-safe transitions, which is actually true for $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$.

Proposition 7.6 *Let $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$. If Γ is weakly $\mathfrak{C}_L$-expanded, then every $rxy \in \Gamma$ is $\mathfrak{C}_L$-safe for Γ.*

Theorem 7.7 (Strong Model Existence) *Let $L \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}\}$. Every open and weakly $\mathfrak{C}_L$-expanded branch is satisfiable within the frame class L.*

One may wish to deal with cases where different accessibility relations underlie different restrictions. This is possible by replacing the variable r in $\mathcal{R}_4$ and $\mathcal{R}_\mathrm{T}$ by fixed parameters representing specific relations. Also, the expandedness condition for boxes will need to distinguish between different relations to match the respective assumptions.

8 Discussion

8.1 The Role of Equality

The tableau rules in Figure 1 clarify the role of equality. If an *equational modality* (D, $\bar{D}$, and $\dot{x}$) is used, the tableau rules will introduce equations. Thus we have modal logic with equality if we use difference modalities or the nominal modality $\dot{x}$ of hybrid logic. Since the formula $\dot{x}y$ is semantically equivalent to an equation $x \doteq y$, one can avoid the introduction of equations and work with formulas $\dot{x}y$. Similar tricks are employed in the prefixed and internalized tableaux of Bolander and Blackburn [6]. Since we rely on classical logic as the underlying logic, where equality is a well-understood primitive, avoiding equations in the syntax would not be helpful. In fact, we see it as an argument in favor of the type-theoretic presentation of modal logic that such syntactic tricks are not needed.

8.2 Alternative Expandedness Conditions

The expandedness condition $\mathcal{E}_D$ for difference formulas differs from the remaining conditions in Section 4 in that, once established for a formula, it is not necessarily preserved by substitution. Consider the branch $\Gamma = \{Dpx, py, x \doteq y\}$. Observe that Dpx is expanded on Γ, so the rule $\mathcal{R}_D$ is not applicable. An application of $\mathcal{R}_{\doteq}$ to Γ yields $\Delta = \{Dpy, py, y \doteq y\}$. But Dpy is no longer expanded on Δ, so $\mathcal{R}_D$ becomes applicable again.

The reader may be wondering why, instead of $\mathcal{E}_D$, we don't take an expandedness condition that corresponds more closely to the tableau rule $\mathcal{R}_D$ and is preserved by substitution:

$$A(D(D(Dp)))x$$
$$\Big|\ \mathcal{R}_A$$
$$D(D(Dp))x$$
$$\Big|\ \mathcal{R}_D$$
$$x \dot{\neq} y,\ D(Dp)y$$
$$\Big|\ \mathcal{R}_D$$
$$y \dot{\neq} z,\ Dpz$$
$$\Big|\ \mathcal{R}_D$$
$$z \dot{\neq} x',\ px'$$
$$\Big|\ \mathcal{R}_A$$
$$D(D(Dp))x'$$
$$\Big|\ \mathcal{R}_D$$
$$\vdots$$

Fig. 3. An Infinite Tableau Derivation using $\mathcal{E}'_D$

($\mathcal{E}'_D$) Dtx is expanded on Γ if, for some y, $x \dot{\neq} y, ty \in \Gamma$

Similarly to $\mathcal{E}_D$, the condition $\mathcal{E}'_D$ suffices to ensure model existence. Unlike $\mathcal{E}_D$, however, $\mathcal{E}'_D$ is not sufficient to ensure termination. Figure 3 shows an infinite tableau derivation that becomes possible if we replace $\mathcal{E}_D$ with $\mathcal{E}'_D$.

Immediately motivated by the the strong model existence theorem is an alternative expandedness condition that one could have taken in place of $\mathcal{E}_\Diamond$:

($\mathcal{E}'_\Diamond$) A formula of the form $\langle r\rangle tx$ is expanded on a branch Γ if there is some y such that $ty \in \Gamma$ and rxy is safe for Γ.

Clearly, $\mathcal{E}'_\Diamond$ does not suffice to achieve termination of our tableau rules. An application of $\mathcal{R}_\Diamond$ to some diamond formula t will, in general, not make t expanded since it does not ensure that the newly added accessibility formula is safe. This allows $\mathcal{R}_\Diamond$ to be applied to t infinitely often. The following alternative to $\mathcal{R}_\Diamond$ remedies the situation:

$$\frac{\langle r\rangle sx \qquad [r]t_1x \qquad \dots \qquad [r]t_nx}{sy,\ t_1y,\ \dots,\ t_ny}\ y \notin \Gamma;\ [r]tx \in \Gamma \text{ implies } t \in \{t_1, \dots, t_n\}$$

Compared to $\mathcal{R}_\Diamond$, this rule seems unnecessarily complex, so we prefer $\mathcal{E}_\Diamond$ over $\mathcal{E}'_\Diamond$.

8.3 Local Substitution

While being a realistic choice for the implementation of a decision procedure, the rule $\mathcal{R}_{\dot{=}}$ is not very convenient for working with paper and pencil. There, one would prefer to just extend branches by new formulas, without ever deleting or modifying formulas that are already there. For this purpose, one can take the following nondestructive and local substitution rule in place of $\mathcal{R}_{\dot{=}}$:

$$\mathcal{R}^n_{\dot{=}} \; \frac{s \qquad t}{t^x_y} \; s \in \{x\dot{=}y, y\dot{=}x\}$$

It is not hard to show that on finite branches, every application of $\mathcal{R}_{\dot{=}}$ resulting in a branch Γ can be simulated by finitely many applications of $\mathcal{R}^n_{\dot{=}}$ that lead to a branch $\Delta \supseteq \Gamma$. Hence, every formula that has a closed tableau in a calculus with $\mathcal{R}_{\dot{=}}$ also has a closed tableau with $\mathcal{R}^n_{\dot{=}}$. In other words, replacing $\mathcal{R}_{\dot{=}}$ with $\mathcal{R}^n_{\dot{=}}$ preserves completeness.

8.4 Nominal versus Prefix Elimination

Our nominal elimination rule is a variant of "nominal substitution" used by van Eijck [27]. Using a suitable representation of variables, nominal elimination can be done in constant time, independently of the size of the branch to which it is applied.

Bolander and Braüner [7] propose a prefixed substitution-based calculus with a different substitution rule:

$$\frac{\sigma x, \tau x \in \Gamma}{\Gamma^\tau_\sigma}$$

where σ, τ are prefixes such that σ is introduced earlier on Γ than τ, and Γ^τ_σ denotes the result of substituting σ for τ on Γ. To distinguish it from nominal elimination as used by our calculus, we refer to this rule as *prefix elimination*. Prefix elimination helps to remove redundancy from branches by collapsing several equivalent prefixes to just one. For instance, consider $\Gamma = \{\sigma\dot{x}, \sigma\langle r\rangle p, \tau\dot{x}, \tau\langle r\rangle q\}$. A naive calculus that does not use any form of elimination would simply add new formulas to Γ, eventually yielding some superset of $\Gamma \cup \{\tau\langle r\rangle p, \sigma\langle r\rangle q\}$. While the approach can be improved to some extent (see [24]), prefix elimination is even more efficient, yielding the branch $\{\sigma\dot{x}, \sigma\langle r\rangle p, \sigma\langle r\rangle q\}$ if we assume that σ is introduced earlier on Γ than τ.

The advantage of nominal elimination compared to prefix elimination is that it allows to remove even more redundancy. Consider the prefixed branch $\Gamma = \{\sigma\dot{x}, \sigma\dot{y}, \sigma\langle r\rangle(@xp), \sigma\langle r\rangle(@yp)\}$ and its counterpart in our calculus $\Delta = \{x\dot{=}z, y\dot{=}z, \langle r\rangle(@xp)z, \langle r\rangle(@yp)z\}$. While prefix elimination is not applicable to Γ, our calculus applied to Δ yields (in two steps) $\{z\dot{=}z, \langle r\rangle(@zp)z\}$.

8.5 Semantic Branching

D'Agostino and Mondadori [9,10] argue convincingly that, from the computational point of view, it is strongly desirable to make different tableau branches semantically disjoint. This optimization, commonly known as semantic branching [20], can be incorporated into our calculus by modifying the two branching rules as follows.

$$\mathcal{R}^b_{\dot{\vee}} \; \frac{(s \mathbin{\dot{\vee}} t)x}{sx \;\mid\; \dot{\neg} sx, tx} \qquad\qquad \mathcal{R}^b_{\bar{D}} \; \frac{\bar{D}tx}{x\dot{=}y \;\mid\; x\dot{\neq}y, ty} \; y \in \mathcal{V}\Gamma$$

The two rules are still sound and preserve completeness with the old expandedness conditions. It is also easy to check that the modifications do not affect termination. Indeed, our termination proof can also be read as a termination proof for the modified calculus.

8.6 Complexity and Caching

In [24], we show that pattern-based blocking can significantly reduce the worst-case size of tableau branches compared to traditional blocking techniques [6,23]. Still, tableau branches can be exponential in the size of the input formula, meaning that a naive implementation of the calculus would have to traverse a double-exponential search space. Caching of satisfiability results for already explored tableau branches is known to reduce the worst-case asymptotic complexity of tableau-based decision procedures for basic modal logic with E to ExpTime [14]. We believe that the same techniques can be extended to deal with nominal equivalence and substitution as used by our procedure, making it optimal for $\mathcal{H}(D)$ over the class of all frames [26,1].

8.7 Related Calculi

Demri [12] presents a sound and complete calculus for a nominal-free logic employing D as the only modal operator, a strictly less expressive fragment of $\mathcal{H}(D)$.

Display calculi for basic hybrid logic with D and inverse modalities are studied by Demri and Goré [13].

Bolander and Blackburn [6] pose the question whether there exists a simple extension of existing tableau calculi for hybrid logic that would enable them to cover the difference modality. Our calculus seems to provide a positive answer to this question. We believe that our treatment of the difference modality can be adapted to most, if not all, of the calculi in [7,6,24].

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