# Assignment 1
# Introduction to Computational Logic, SS 2005

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Read in the lecture notes: the chapter on type theory

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**General Information:** Every week we will give you a new assignment sheet with exercises for the lecture. Assignment sheets do not have to be handed in, and will not be graded. But we will provide sample solutions that you can compare with your own solutions. These sample solutions will appear one week after the corresponding assignment sheet.

**Remark:** If not specified otherwise, we assume the following constants:

$$\begin{array}{rcl} 0, 1 & : & \mathbb{B} \\ \wedge, \vee, \Rightarrow, \Leftrightarrow & : & \mathbb{B} \to \mathbb{B} \to \mathbb{B} \\ \doteq_T & : & T \to T \to \mathbb{B} \\ \forall_T, \exists_T & : & (T \to \mathbb{B}) \to \mathbb{B} \end{array}$$

**Exercise 1.1** Let the following logical constants be given:

$$0 : \mathbb{B} \qquad \Rightarrow : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$$

Express the following values with these constants. (Once a value has been defined, you can use it for defining the others.)

a) $\neg \in \mathbb{B} \to \mathbb{B}$

b) $1 \in \mathbb{B}$

c) $\vee \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

d) $\wedge \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

e) $\Leftrightarrow \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

f) $\Leftarrow \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

g) $\doteq \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

**Exercise 1.2 (Sets)** Let $X$ be a set. Then the functions $X \to \mathbb{B}$ represent the subsets of $X$. We write $P(X)$ as abbreviation for the type $X \to \mathbb{B}$. Express the following set operations with the help of the logical constants:

a) Intersection $\cap \colon P(X) \to P(X) \to P(X)$;

b) Union $\cup \colon P(X) \to P(X) \to P(X)$;

c) Difference $- \colon P(X) \to P(X) \to P(X)$;

d) Test for subset $\subseteq \colon P(X) \to P(X) \to \mathbb{B}$;

e) Test for disjointness $\| \colon P(X) \to P(X) \to \mathbb{B}$;

f) Test for membership $\in \colon X \to P(X) \to \mathbb{B}$.

**Exercise 1.3** Let the following constants be given:

$$\begin{aligned} \neg &: \mathbb{B} \to \mathbb{B} \\ \wedge &: \mathbb{B} \to \mathbb{B} \to \mathbb{B} \\ \exists &: (\mathbb{N} \to \mathbb{B}) \to \mathbb{B} \\ + &: \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\ \cdot &: \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\ \leq &: \mathbb{N} \to \mathbb{N} \to \mathbb{B} \end{aligned}$$

Express the following values with these constants. (Once a value has been defined you can use it under the given name.)

a) Equality $= \in \mathbb{N} \to \mathbb{N} \to \mathbb{B}$;

b) Less-than test $< \in \mathbb{N} \to \mathbb{N} \to \mathbb{B}$;

c) Disjunction $\vee \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$;

d) Implication $\Rightarrow \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$;

e) Universal quantification $\forall \in (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$;

f) A function *min* $\in \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{B}$, which for $x$, $y$, $z$ gives 1, if and only if $z$ is the minimum of $x$ and $y$;

g) A function *one* $\in \mathbb{N} \to \mathbb{B}$, which for $x$ gives 1, if and only if $x = 1$;

h) A function *divide* $\in \mathbb{N} \to \mathbb{N} \to \mathbb{B}$, which for $x$, $y$ gives 1, if and only if $x$ is a factor of $y$;

i) A function *mod* $\in \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{B}$, which for $x$, $y$, $z$ gives 1, if and only if $z$ is the remainder of the division of $x$ by $y$;

j) A function *gcd* $\in \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{B}$, which for $x$, $y$, $z$ gives 1, if and only if $z$ is the greatest common divisor of $x$ and $y$;

k) A function *bound* $\in (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$, which tests whether its argument has an upper bound in $\mathbb{N}$;

l) A function *max* $\in (\mathbb{N} \to \mathbb{B}) \to \mathbb{N} \to \mathbb{B}$, which for $f$ and $x$ gives 1, if and only if $f$ is a set with an upper bound in $\mathbb{N}$ and $x$ is its greatest element.

**Exercise 1.4 (Schönfinkels U-combinator)** Let $X$ be a non-empty set and let the so-called U-combinator for $X$ be defined as follows:

$$U \in (X \to \mathbb{B}) \to (X \to \mathbb{B}) \to \mathbb{B}$$
$$U f g = \forall x \in X \,.\, \neg(f x \wedge g x)$$

If $f$ and $g$ are interpreted as subsets of $X$, then $U$ tests whether $f$ and $g$ are disjoint. Describe the following functions with $U$:

a) Negation $\neg \in \mathbb{B} \to \mathbb{B}$;

b) Conjunction $\wedge \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$;

c) Universal quantification $\forall \in (X \to \mathbb{B}) \to \mathbb{B}$;

d) Existential quantification $\exists \in (X \to \mathbb{B}) \to \mathbb{B}$.

**Exercise 1.5 (Choice Operator)** Let the following constants be given:

$$\begin{aligned}
\neg &: \mathbb{B} \to \mathbb{B} \\
\wedge &: \mathbb{B} \to \mathbb{B} \to \mathbb{B} \\
\vee &: \mathbb{B} \to \mathbb{B} \to \mathbb{B} \\
+ &: \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
\cdot &: \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
\leq &: \mathbb{N} \to \mathbb{N} \to \mathbb{B} \\
= &: \mathbb{N} \to \mathbb{N} \to \mathbb{B}
\end{aligned}$$

Additionally let the so-called *choice operator* $\epsilon : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$ be given, which for every non-empty subset $M$ of $\mathbb{N}$ gives an element of $M$. Formally speaking:

$$\forall f \in \mathbb{N} \to \mathbb{B} \,.\, (\exists x \in \mathbb{N} \,.\, f x) \Longrightarrow f(\epsilon f)$$

Describe the following values with help of the given constants. (As before, once a value is defined, you can use it.)

a) $0 \in \mathbb{N}$;

b) Existential quantifier $\exists \in (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$;

c) $1 \in \mathbb{N}$;

d) Subtraction $- \in \mathbb{N} \to \mathbb{N} \to \mathbb{N}$;

e) A function *if* $\in \mathbb{B} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N}$, which gives for $b$, $x$, $y$ depending on $b$ the value $x$ or $y$ ($x$ if $b = 1$, otherwise $y$);

f) A function *max* $\in \mathbb{N} \to \mathbb{N} \to \mathbb{N}$, which gives the maximum of two numbers;

g) A function $d \in \mathbb{N} \to \mathbb{N} \to \mathbb{N}$, which gives for $x$, $y$ the result of the integer division of $x$ by $y$ if $y > 0$. (Example: $d(7)(3) = 2$.)

**Exercise 1.6** Consider the following terms:

a) $\forall x \,.\, \forall y \,.\, (x \doteq y) \Leftrightarrow \forall f \,.\, fx \Rightarrow fy$ where $x\colon X$.

b) $\forall f \,.\, \forall g \,.\, (f \doteq g) \Leftrightarrow \forall x \,.\, fx \doteq gx$ where $f, g\colon X \to Y$.

c) $\forall x \,.\, \forall y \,.\, (x \wedge y) \Leftrightarrow \forall f \,.\, fxy \Rightarrow f11$ where $1\colon \mathbb{B}$.

For each term do the following

- give all variables and their types,
- give all constants and their types,
- draw the De Bruijn tree representation,
- give the value that the term denotes.

**Exercise 1.7 (Normal Forms)** Simplify the following terms with the $\beta$- and the $\eta$-rule:

a) $(\lambda z.a)(\lambda xy.fxy)$

b) $(\lambda xu.x)(\lambda vy.y)$

c) $(\lambda xy.fxy)ab$

d) $(\lambda f.f(a))(\lambda xy.fxy)b$

e) $(\lambda z.(\lambda xy.fxy)((\lambda z.z)z))ab$

f) $(\lambda fgx.fx(gx))(\lambda xy.x)(\lambda xz.x)$

**Exercise 1.8 (Substitution)** Apply the following substitutions:

a) $(\lambda x.yx)[y := x]$

b) $y(\lambda x.xy)[y := x]$

c) $(\lambda x \lambda y.fxy)[f := xy]$