Assignment 4
Introduction to Computational Logic, SS 2005

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http://www.ps.uni-sb.de/courses/cl-ss05/

Boolean Equations

Let BA (Boolean Axioms) be the following set of equations:

Commutativity:	$xy = yx$	$x + y = y + x$
Assocativity:	$(xy)z = x(yz)$	$(x + y) + z = x + (y + z)$
Distributivity:	$x(y + z) = xy + xz$	$x + yz = (x + y)(x + z)$
Complements:	$x\bar{x} = 0$	$x + \bar{x} = 1$
Identities:	$1x = x$	$0 + x = x$

You will show that the following equations can be deduced from BA.

Idempotence:	$xx = x$	$x + x = x$
Dominance:	$0x = 0$	$1 + x = 1$
Absorption:	$x(x + y) = x$	$x + xy = x$
Resolution:	$xy + \bar{x}z = xy + \bar{x}z + yz$	$(x + y)(\bar{x} + z) = (x + y)(\bar{x} + z)(y + z)$
Double negation:	$\bar{\bar{x}} = x$	$\bar{\bar{x}} = x$
Dual identities:	$\bar{1} = 0$	$\bar{0} = 1$
deMorgan:	$\overline{xy} = \bar{x} + \bar{y}$	$\overline{x + y} = \bar{x}\bar{y}$

We use BL (Boolean Laws) as name for the set of equations mentioned above.

Exercise 4.1 Prove the following:

a) BA $\vdash xx = x$
Hint: start with $xx = xx + 0$ and then use complements

b) BA $\vdash 0x = 0$
Hint: start with $0x = 0x + 0$ and then use complements

c) BA $\vdash x = x(x + y)$
Hint: start with $x = x + 0$ and then use (b)

d) BA $\vdash xy + \bar{x}z = xy + \bar{x}z + yz$
Hint: start from left by using (c) (in the form of $x = x(x+y)$ and $\bar{x} = \bar{x}(\bar{x}+y)$); later use (a)

Exercise 4.2 (Uniqueness of Complements) Prove the following:

$$\{xy = 0, x + y = 1\} \overset{\mathsf{BA}}{\vdash\dashv_\circ} \{y = \bar{x}\}$$

Hint: The direction $\dashv_\circ$ is easy. For $\vdash_\circ$ start from y and then use $y = 1y$ and complements.

Exercise 4.3 Prove the following:

a) $\mathsf{BA} \vdash x = \bar{\bar{x}}$

b) $\mathsf{BA} \vdash 0 = \bar{1}$

c) $\mathsf{BA} \vdash \bar{x} + \bar{y} = \overline{xy}$

Hint: In each case use an instance of the conservative deductive equivalence from Exercise 4.2.

Exercise 4.4 (Conservative Deduction) Prove the following:

a) $\{x = 1, y = 1\} \overset{\mathsf{BL}}{\vdash\dashv_\circ} \{xy = 1\}$

b) $\{x = 0, y = 0\} \overset{\mathsf{BL}}{\vdash\dashv_\circ} \{x + y = 0\}$

c) $\{x \Rightarrow y = 1\} \overset{\mathsf{BL}}{\vdash\dashv_\circ} \{x = xy\}$

d) $\{x = y\} \overset{\mathsf{BL}}{\vdash\dashv_\circ} \{x \Leftrightarrow y = 1\}$

e) $\{x = xy\} \overset{\mathsf{BL}}{\vdash\dashv_\circ} \{y = y + x\}$

Exercise 4.5 Prove the following:

a) $\{0 = 1\} \overset{\mathsf{BL}}{\vdash_\circ} \{x = y\}$

b) $\{x = 1, x \Rightarrow y = 1\} \overset{\mathsf{BL}}{\vdash_\circ} \{y = 1\}$ (Modus Ponens)

Exercise 4.6 Find an equation e such that $\mathrm{Sol}_{\mathbb{N}}\{e\} \neq \mathrm{Sol}_{\mathbb{Z}}\{e\}$. You can use the constants $+, -, \cdot, 0, 1$ and the variables x, y.