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http://www.ps.uni-sb.de/courses/cl-ss05/

Exercise 7.1 (Sequent Calculus and Peirce's Law) Prove the sequent

$$\emptyset \Rightarrow ((x \to y) \to x) \to x$$

in the symmetric sequent calculus used for the completeness proof of ND.

Exercise 7.2 (Combinatorial Proofs) Consider the propositional type $t = X \to X$.

a) Find two different closed $\beta\eta$-normal proof term for t.

b) Find an ND proof for t.

c) Find a closed combinatorial proof term for t.

d) Find a combinatorial proof for t.

Exercise 7.3 (ND-Proofs) Consider now the Boolean term $t = ((x \to 0) \to x) \to x$.

a) Show by rewriting with respect to BL that t is valid.

b) Find a closed proof term for the propositional type corresponding to t.

c) Find an ND-proof for t.

d) The validity of t means that we can assume $\bar{s}$ when we try to prove s. We can formulate this insight with the ND-proof rule

$$\frac{C, s \to 0 \Rightarrow s}{C \Rightarrow s}$$

Show that this rule can be simulated with the rules defining $\overset{\mathsf{N}}{\vdash}$.

Exercise 7.4 (Church numerals) Consider the propositional type

$$t = (X \to X) \to X \to X$$

We call a proof term *nice* if it is closed, $\beta\eta$-normal, and does not contain δ.

a) Give nice proof terms of type t. Explain how the nice proof terms of type t correspond to the natural numbers.

b) Convince yourself that $\beta\eta$-reduction transforms an application pq of two nice proof terms p, q into a nice proof term.

c) Give a nice proof term of type $t \to t \to t$ that corresponds to addition of natural numbers.

Exercise 7.5 (Proof Terms) Instead of the constants

$$\delta_t : \bar{\bar{t}} \to t$$

one can also have the constants

$$\phi_{s,t} : (\bar{s} \to \bar{t}) \to (\bar{s} \to t) \to s$$

where $\bar{t}$ abbreviates $t \to 0$.

a) Find a closed proof term for $\tau(\phi_{s,t})$ that does not use ϕ. (Recall that $\tau(p)$ is the type of p.)

b) (Challenge!) Find a closed proof term for $\tau(\delta_t)$ that does not use δ but uses ϕ.

Exercise 7.6 (Properties of $\overset{\mathsf{N}}{\vdash}$) We consider sequents that may contain the constants $\to$ and 0. Prove the following:

a) $C, s \Rightarrow t \quad \overset{\mathsf{N}}{\vdash\dashv} \quad C \Rightarrow s \to t$

b) $\forall C.(C \Rightarrow s \overset{\mathsf{N}}{\vdash} C \Rightarrow t) \qquad \text{iff} \qquad \overset{\mathsf{N}}{\vdash} s \Rightarrow t$

Exercise 7.7 (Higher-order Boolean Logic) Prove the following:

a) $\mathsf{HB} \vdash x \wedge fx = x \wedge f1$

b) $\mathsf{HB} \vdash \bar{x} \wedge fx = \bar{x} \wedge f0$

c) $\mathsf{HB} \vdash fx = \bar{x} \wedge f0 + x \wedge f1$

d) $\mathsf{HB} \vdash f0 \wedge f1 \to fx = 1$

e) For all s, t of type B:

$$\mathsf{HB} \vdash s = t \quad \Longleftrightarrow \quad \begin{array}{l} \mathsf{HB} \vdash s[x := 0] = t[x := 0] \\ \wedge\ \mathsf{HB} \vdash s[x := 1] = t[x := 1] \end{array}$$

Recall that $\mathsf{HB} = \mathsf{BA} \cup \{\mathsf{BRep}\}$, where BRep is

$$x \leftrightarrow y \wedge fx \quad = \quad x \leftrightarrow y \wedge fy$$

Exercise 7.8 (Higher-order Boolean Logic) Let A be a set of equations such that $\mathsf{BA} \subseteq A$ and for all s, t of type B:

$$s = 1 \overset{A}{\vdash}_{\circ} t = 1 \quad \Rightarrow \quad A \vdash s \to t = 1$$

Prove $A \vdash x \leftrightarrow y \wedge fx \to fy = 1$. (Recall the operator precedence in the lecture notes.)

Exercise 7.9 (Higher-order Boolean Logic) Let $\mathcal{D}$ be an interpretation that satisfies BA and also $fx = \bar{x} \wedge f0 + x \wedge f1$ (f and x are variables with types $B \to B$ and B, respectively). Prove $\mathcal{D}B = \{\mathcal{D}0, \mathcal{D}1\}$.