Quick Reference Guide to S

Definitions

$E \overset{A}{\vDash}_\circ F \quad \overset{\text{def}}{\iff} \quad$ for every interpretation satisfying A: every solution of E is a solution of F

$Beta \overset{\text{def}}{=} \{ (\lambda x.\, s)t = s[x{:=}\, t] \mid (\lambda x.\, s)t \in Ter \}$

$Eta \overset{\text{def}}{=} \{ \lambda x.\, sx = s \mid sx \in Ter$ and $x \notin FV(s) \}$

$Lam \overset{\text{def}}{=} Beta \cup Eta$

$Rep\ E \overset{\text{def}}{=} \{ s = s' \mid \exists\, t = t' \in E\colon\ s$ is obtainable from s' by replacing a subterm t' with $t \}$

$Rep_\circ\ E \overset{\text{def}}{=} \{ s = s' \mid \exists\, t = t' \in E\colon\ s$ is obtainable from s' by replacing a subterm t' with t such that no free variable of $t = t'$ is captured $\}$

$Sub\ E \overset{\text{def}}{=} \{ S\theta s = S\theta t \mid s = t \in E$ and θ substitution $\}$

$\overleftrightarrow{E} \overset{\text{def}}{=} E \cup E^{-1}$

$Con_\circ\ E \overset{\text{def}}{=} Rep_\circ\ \overleftrightarrow{E}$

$Con\ E \overset{\text{def}}{=} Rep\,(Sub\ \overleftrightarrow{E})$

$E \overset{A}{\vdash}_\circ F \quad \overset{\text{def}}{\iff} \quad F \subseteq (Con\, Lam \cup Con_\circ\ E \cup Con\ A)^*$

$E \overset{A}{\vdash\!\dashv}_\circ F \quad \overset{\text{def}}{\iff} \quad E \overset{A}{\vdash}_\circ F \wedge F \overset{A}{\vdash}_\circ E$

$E \overset{A}{\vdash} F \quad \overset{\text{def}}{\iff} \quad \overset{E\cup A}{\vdash}_\circ F$

$E \overset{A}{\vdash\!\dashv} F \quad \overset{\text{def}}{\iff} \quad E \overset{A}{\vdash} F \wedge F \overset{A}{\vdash} E$

Omission of E and A means that they are empty. Presence of $\circ$ means that the variables in E and F are parametric and the variables in A are universal. If $\circ$ is not present, all variables are universal.

Structural Properties

Soundness $E \overset{A}{\vdash}_\circ F \Longrightarrow E \overset{A}{\models}_\circ F$

Expansivity $E \overset{A}{\vdash}_\circ E \cup A$

Accumulation $E \overset{A}{\vdash}_\circ F_1 \wedge E \overset{A}{\vdash}_\circ F_2 \Longrightarrow E \overset{A}{\vdash}_\circ F_1 \cup F_2$

Monotonicity $E \overset{A}{\vdash}_\circ F \wedge E \subseteq E' \wedge A \subseteq A' \wedge F' \subseteq F \Longrightarrow E' \overset{A'}{\vdash}_\circ F'$

Transitivity $E \overset{A}{\vdash}_\circ E' \wedge E' \overset{A}{\vdash}_\circ F \Longrightarrow E \overset{A}{\vdash}_\circ F$

$\overset{A}{\vdash}_\circ A' \wedge E \overset{A'}{\vdash}_\circ F \Longrightarrow E \overset{A}{\vdash}_\circ F$

Duality $E \overset{A}{\vdash}_\circ F \Longrightarrow \hat{E} \overset{A}{\vdash}_\circ \hat{F}$ if $A \vdash \hat{A}$

where $\hat{}$ is a type preserving and self inverting function mapping term constants to term constants

Compactness $E \overset{A}{\vdash}_\circ e \Longrightarrow \exists$ finite $E' \subseteq E$ $\exists$ finite $A' \subseteq A$: $E' \overset{A'}{\vdash}_\circ e$

Semi-decidability E and A semi-decidable $\Longrightarrow \{\, e \mid E \overset{A}{\vdash}_\circ e \,\}$ semi-decidable

SuS $E \overset{A}{\vdash}_\circ F \Longrightarrow S\theta E \overset{A}{\vdash}_\circ S\theta F$ Stability under Substitution

Conversion $E \overset{A}{\vdash}_\circ F \iff F \subseteq (\mathit{Con}\,\mathit{Lam} \cup \mathit{Con}_\circ\, E \cup \mathit{Con}\, A)^*$

$E \overset{A}{\vdash}_\circ F \Longrightarrow E \overset{A}{\vdash}_\circ (\mathit{Con}_\circ\, F)^*$

$E \overset{A}{\vdash}_\circ F \Longrightarrow E \overset{A}{\vdash}_\circ (\mathit{Con}\, F)^*$ if $FV(E) \cap FV(F) = \emptyset$

EE $sx = tx \vdash\!\dashv s = t$ if $x \notin FV(s,t)$ External Extensionality

Xi $s = t \vdash\!\dashv \lambda x.\, s = \lambda x.\, t$

Quick Reference Guide to BA, HB, BQ, and HOL

Logical Constants $0, 1 : B$; $\neg : B \to B$; $\vee, \wedge : B \to B \to B$

$\forall_T, \exists_T : (T \to B) \to B$; $\mathbf{C}_T : (T \to B) \to T$

Duality $1 \leftrightsquigarrow 0$ $\wedge \leftrightsquigarrow \vee$ $\forall \leftrightsquigarrow \exists$

Notations

$$\overline{x} \stackrel{\text{def}}{=} \neg x$$

$$x \to y \stackrel{\text{def}}{=} \overline{x} \vee y$$

$$x \leftrightarrow y \stackrel{\text{def}}{=} x \wedge y \vee \overline{x} \wedge \overline{y}$$

$$\forall x. s \stackrel{\text{def}}{=} \forall (\lambda x. s)$$

$$x \doteq y \stackrel{\text{def}}{=} \forall f. fx \to fy$$

Operator Precedence $\doteq$, $\leftrightarrow$, $\wedge$, $\vee$, $\to$

Formula Convention write s for $s = 1$

Axioms

BA is the set containing the following equations and their duals:

Commutativity $x \wedge y = y \wedge x$

Associativity $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

Distributivity $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$

Identity $1 \wedge x = x$

Complement $x \wedge \overline{x} = 0$

HB is the set obtained from *BA* by adding the following equation:

BRep $x \leftrightarrow y \wedge fx \;=\; x \leftrightarrow y \wedge fy$ *Boolean Replacement*

BQ is the set obtained from *HB* by adding the following equations and their duals for all types:

$\forall 1$ $\forall x. 1 = 1$

$\forall I$ $\forall f = \forall f \wedge fx$ *Instantiation*

HOL is the set obtained from BQ by adding the following equations for all types:

Ext $\forall x. fx \doteq gx \;=\; f \doteq g$ *Extensionality*

Choice $\exists f = f(\mathbf{C}f)$

Structural Properties

And $x, y \overset{BA}{\vdash\!\dashv_\circ} x \wedge y$

Equi $x = y \overset{BA}{\vdash\!\dashv_\circ} x \leftrightarrow y$

Ded $s \overset{A}{\vdash\!\dashv_\circ} t \iff A \vdash s = t$ if $A \vdash HB$ *Deductivity*

$s \overset{A}{\vdash_\circ} t \iff A \vdash s \to t$ if $A \vdash HB$

$E, s \overset{A}{\vdash_\circ} t \iff E \overset{A}{\vdash_\circ} s \to t$ if $A \vdash HB$

Gen $s \overset{BQ}{\vdash\!\dashv} \forall x.s$ *Generalisation*

$E \overset{A}{\vdash_\circ} s \iff E \overset{A}{\vdash_\circ} \forall x.s$ if $x \notin FV(E)$ and $A \vdash BQ$

Duality BA, HB, BQ and $BQ \cup Ext$ satisfy duality (i.e., $A \vdash \widehat{A}$)

BDual $s \overset{A}{\vdash_\circ} t \iff \hat{t} \overset{A}{\vdash_\circ} \hat{s}$ if $A \vdash HB \cup \widehat{A}$

Equational Laws for BA

Idem $x \wedge x \;=\; x$ $x \vee x \;=\; x$ *Idempotence*

Dom $0 \wedge x \;=\; 0$ $1 \vee x \;=\; 1$ *Dominance*

Abs $x \wedge (x \vee y) \;=\; x$ $x \vee x \wedge y \;=\; x$ *Absorption*

Res $(x \wedge y) \vee (\overline{x} \wedge z) \;=\; (x \wedge y) \vee (\overline{x} \wedge z) \vee (y \wedge z)$
$(x \vee y) \wedge (\overline{x} \vee z) \;=\; (x \vee y) \wedge (\overline{x} \vee z) \wedge (y \vee z)$ *Resolution*

dM $\overline{x \wedge y} \;=\; \overline{x} \vee \overline{y}$ $\overline{x \vee y} \;=\; \overline{x} \wedge \overline{y}$ *de Morgan*

NegId $\overline{1} \;=\; 0$ $\overline{0} \;=\; 1$ *Negated Identities*

DNeg $\overline{\overline{x}} \;=\; x$ *Double Negation*

Equational Laws for →

Contra $x \to y \;=\; \overline{y} \to \overline{x}$ *Contraposition*

SF $x \to y \to z \;=\; x \wedge y \to z$ *Schönfinkel*

MP $x \wedge (x \to y) \;=\; x \wedge y$ *Modus ponens*

T $x \wedge y \to x \vee z$ *Triviality*

Ref $x \to x$ *Reflexivity*

W $(x \to y) \to x \wedge x' \to y \qquad (x \to y) \to x \to y \vee y'$ *Weakening*

Trans $(x \to y) \to (y \to z) \to x \to z$ *Transitivity*

Mon $(x \to y) \to x \wedge z \to y \wedge z \qquad (x \to y) \to x \vee z \to y \vee z$ *Monotonicity*

$\to 0$ $x \to 0 \;=\; \overline{x}$

$1 \to$ $1 \to x \;=\; x$

$0 \to$ $0 \to x \;=\; 1$

$\to 1$ $x \to 1 \;=\; 1$

$\neg \to$ $x \wedge \overline{y} \to z \;=\; x \to y \vee z$

$\to \neg$ $x \to y \vee \overline{z} \;=\; x \wedge z \to y$

$\vee \to$ $x \vee y \to z \;=\; (x \to z) \wedge (y \to z)$

$\to \wedge$ $x \to y \wedge z \;=\; (x \to y) \wedge (x \to z)$

Equational Laws for ↔

Comm $x \leftrightarrow y \;=\; y \leftrightarrow x$ *Commutativity*

Ass $(x \leftrightarrow y) \leftrightarrow z \;=\; x \leftrightarrow (y \leftrightarrow z)$ *Associativity*

GR $x \to y \;=\; x \leftrightarrow (x \wedge y) \;=\; y \leftrightarrow (y \vee x)$ *Golden Rule*

UoC $x \leftrightarrow \overline{y} \;=\; (x \vee y) \wedge \overline{x \wedge y}$ *Uniqueness of Complements*

MI $x \leftrightarrow y \;=\; (x \to y) \wedge (y \to x)$ *Mutual Implication*

Ref $x \leftrightarrow x = 1$ *Reflexivity*

Neg $x \leftrightarrow y = \overline{x} \leftrightarrow \overline{y}$ *Negation*

dM $\overline{x \leftrightarrow y} = x \leftrightarrow \overline{y}$ *de Morgan*

$\leftrightarrow 1$ $x \leftrightarrow 1 = x$

$\leftrightarrow 0$ $x \leftrightarrow 0 = \overline{x}$

Trading $x \to (y \leftrightarrow z) = x \wedge y \leftrightarrow x \wedge z = (x \wedge y \to z) \wedge (x \wedge z \to y)$

Equational Laws for HB

BRep $x \leftrightarrow y \wedge fx = x \leftrightarrow y \wedge fy$ *Boolean Replacement*

BExp $fx = \overline{x} \wedge f0 \vee x \wedge f1$ *Boolean Expansion*

BCA $f0 \wedge f1 \to fx$ *Boolean Case Analysis*

BRep, BExp and BCA are $\overset{BA}{\mapstochar\!\dashv}$- equivalent.

Equational Laws for BQ

$\forall E$ $\forall x. q = q$ *Elimination*

$\forall \vee$ $\forall f \vee q = \forall x. fx \vee q$

$\forall \wedge$ $\forall f \wedge q = \forall x. fx \wedge q$

$\forall \wedge'$ $\forall f \wedge \forall g = \forall x. fx \wedge gx$

dM $\overline{\forall f} = \exists x. \overline{fx}$ *de Morgan*

$\forall\forall$ $\forall x. \forall y. fxy = \forall y. \forall x. fxy$

$\forall B$ $\forall f = f0 \wedge f1$

$\forall\to$ $\forall f \to q = \exists x. fx \to q$

$\exists\to$ $\exists f \to q = \forall x. fx \to q$

$\to\forall$ $q \to \forall f = \forall x. q \to fx$

$\to\exists$ $q \to \exists f = \exists x. q \to fx$

$\forall\to\exists$ $\forall f \to \exists g = \exists x. fx \to gx$

Ref $x \doteq x$

Sym $x \doteq y \; = \; y \doteq x$

Tra $x \doteq y \wedge y \doteq z \to x \doteq z$

Rep $x \doteq y \wedge fx \; = \; x \doteq y \wedge fy$ *Replacement*

BIA $x \doteq y \; = \; x \leftrightarrow y$ *Boolean Identity Agrees*

Equational Laws for BQ + Ext

IXi $\forall x. s \doteq t \; = \; (\lambda x. s) \doteq (\lambda x. t)$ *Internal ξ*

$\forall D$ $\forall f \; = \; f \doteq (\lambda x. 1)$

$\exists D$ $\exists f \; = \; \overline{f \doteq (\lambda x. 0)}$

Equational Laws for BQ + Choice

Sko $\forall x. \exists y. fxy \; = \; \exists h. \forall x. fx(hx)$ *Skolem*