Closure Operators and Entailment Relations

10-1 G. Smolka June 13, 2005

Abstract View of Logic Reasoning

- Assumptions $\rightsquigarrow$ Consequences
- Judgements: $x \in \mathcal{X}$ syntactic objects, e.g., $s=t$
- Closure operator: $f \in \mathcal{P}\mathcal{X} \to \mathcal{P}\mathcal{X}$
  assumptions $\to$ consequences
- Entailment relation: $x \in fA$

Closure Operators

Set $\mathcal{X}$

$f \in \mathcal{P}\mathcal{X} \to \mathcal{P}\mathcal{X}$ **closure operator on $\mathcal{X}$** if for all $A, A' \subseteq \mathcal{X}$:

- $A \subseteq fA$ — Expansivity
- $A \subseteq A' \Rightarrow fA \subseteq fA'$ — Monotonicity
- $f(fA) = fA$ — Idempotence

fA: closure of A

Entailment Relations

Set $\mathcal{X}$

$\vdash \subseteq \mathcal{P}\mathcal{X} \times \mathcal{X}$ **entailment relation on $\mathcal{X}$** if

$\exists$ closure operator f on $\mathcal{X}$ $\forall A \subseteq \mathcal{X}$ $\forall x \in \mathcal{X}$.

$$A \vdash x \iff x \in fA$$

closure operator $\longleftrightarrow$ entailment relation

Pronunciation:

$\vdash$ turnstile

$A \vdash x$ A entails x

Notations

$x_1, \dots, x_n \vdash x \overset{def}{\iff} \{x_1, \dots, x_n\} \vdash x$

$\vdash x \overset{def}{\iff} \emptyset \vdash x$

$A \vdash B \overset{def}{\iff} \forall b \in B.\ A \vdash b$ — reflexive, transitive

$A \dashv\vdash B \overset{def}{\iff} A \vdash B \wedge B \vdash A$ — equivalence relation

$\mathcal{C}(\vdash) \overset{def}{=} \lambda A \subseteq \mathcal{X}.\ \{x \mid A \vdash x\}$ — associated closure operator

Direct Characterization of Entailment Relations

Set $\mathcal{X}$

$\vdash \subseteq \wp\mathcal{X} \times \mathcal{X}$ entailment relation on $\mathcal{X}$

iff for all $A, A' \subseteq \mathcal{X}$:

- $x \in A \Rightarrow A \vdash x$ — Expansivity
- $A \vdash x \wedge A \subseteq A' \subseteq \mathcal{X} \Rightarrow A' \vdash x$ — Monotonicity
- $A \vdash B \wedge A \cup B \vdash x \Rightarrow A \vdash x$ — Idempotence

Properties

If f closure operator on $\mathcal{X}$ and $A, B, C \subseteq \mathcal{X}$:

1) $f(A \cup B) = fA \iff B \subseteq fA$

2) $f(A \cup B) = f(A \cup C) \iff B \subseteq f(A \cup C) \wedge C \subseteq f(A \cup B)$

3) $fA = fB \Rightarrow f(A \cup C) = f(B \cup C)$

4) $f(A \cup B) \Rightarrow f(fA \cup B)$

The composition of closure operators on $\mathcal{X}$ is a closure operator on $\mathcal{X}$

Soundness

f, g, h closure operators on $\mathcal{X}$

$f \leq g \overset{def}{\iff} \forall A \subseteq \mathcal{X}.\ fA \subseteq gA$ — f sound for g

"$f \leq g$" is partial order

If $\vdash, \vdash'$ entailment relations on $\mathcal{X}$:
$\mathcal{C}(\vdash) \leq \mathcal{C}(\vdash') \iff \vdash \subseteq \vdash'$

If $f \leq g$, then $\forall A \subseteq \mathcal{X}$: $fA \cup f(gA) \cup g(fA) \subseteq gA$

$f, g \leq h \Rightarrow f \circ h \leq h$

Compactness

f closure operator on $\mathcal{X}$

f compact if $\forall A \subseteq \mathcal{X}\ \forall x \in \mathcal{X}.$

$$x \in fA \Rightarrow \exists \text{ finite } A' \subseteq A.\ x \in fA'$$

Compactness means that
every consequence can be obtained from
finitely many assumptions

Computational Closure Operators

f closure operator on $\mathcal{X}$

f computational if f compact and

$\forall A \subseteq \mathcal{X}.\quad A$ semi-decidable $\Rightarrow fA$ semi-decidable

The composition of computational closure operators on $\mathcal{X}$
is a computational closure operator on $\mathcal{X}$