# Formula Convention and Linear Proofs

10-3 G. Smolka June 15, 2005

# Formula Convention

A **formula** is a term of type $\mathbb{B}$.

$\boxed{\text{An equation } s = 1 \text{ may be written as } s}$

This is just notational sugar.

It greatly improves the readability of entailment laws and proofs, as you will see from the following examples.

# Entailment Laws for BA

| | |
|---|---|
| Sub | $s \vdash s[x:=t]$ |
| Alpha | $s \dashv\vdash s[x:=y]$ |
| And | $x \wedge y \;\dashv\vdash^{BA}_{0}\; x, y$ |
| Neg | $x = y \;\dashv\vdash^{BA}_{0}\; \bar{x} = \bar{y}$ |
| Or | $x \;\vdash^{BA}_{0}\; x \vee y$ |
| Equi | $x \Leftrightarrow y \;\dashv\vdash^{BA}_{0}\; x = y \;\dashv\vdash^{BA}_{0}\; x \to y,\; y \to x$ |
| MP | $x \to y,\; x \;\vdash^{BA}_{0}\; y$ |
| GR | $x \to y \;\dashv\vdash^{BA}_{0}\; x = x \wedge y \;\dashv\vdash^{BA}_{0}\; y = y \vee x$ |
| DoC | $x = \bar{y} \;\dashv\vdash^{BA}_{0}\; x \vee y,\; \overline{x \wedge y}$ |

# Entailment Laws for HB and Duality

| | | |
|---|---|---|
| BCA | $s0,\; s1 \dashv\vdash sx$ | |
| Ded | $A \vdash s \to t \iff s \vdash^{A}_{0} t$ | if $A \vdash HB$ |
| | $A \vdash s = t \iff s \dashv\vdash^{A}_{0} t$ | if $A \vdash HB$ |
| Dual | $s \to t \;\dashv\vdash^{A}\; \hat{t} \to \hat{s}$ | if $\exists A'.\; A' \dashv\vdash A \;\wedge\; \hat{A'} = A' \;\wedge\; A \vdash BA$ |

# Entailment Laws for BQ

| | |
|---|---|
| $\forall I$ | $\forall f \vdash^{BQ}_{0} fx$ |
| $\exists I$ | $fx \vdash^{BQ}_{0} \exists f$ |
| Gen | $s \dashv\vdash^{BQ} \forall x.s$ |

# Linear Proofs

**Conversion Proof**: Shows $E \vdash^{A}_{0} s_1 = s_n$ by a sequence $s_1 = s_2 = s_3 = \cdots = s_n$ where each step $s_i = s_{i+1}$ represents one or several conversions.

formerly called rewriting proofs

We have seen many examples

**Entailment Proof**: Shows $E_n \vdash^{A} E_1$ by a sequence $E_1 \dashv E_2 \dashv E_3 \dashv \cdots \dashv E_n$ where $E_{i+1} \vdash^{A} E_i$ for each step by simple arguments. Can mix in $E_i \dashv\vdash E_{i+1}$ and conversion steps.

new!

# Example: Linear Entailment Proof for Cantor's Law

Claim: $BQ \vdash \overline{\exists f \forall g \exists x \forall y.\ fxy \Leftrightarrow gy}$

Proof:

| | | |
|---|---|---|
| | $\overline{\exists f \forall g \exists x \forall y.\ fxy \Leftrightarrow gy}$ | |
| $=$ | $\forall f \exists g \forall x \exists y.\ \overline{fxy} \Leftrightarrow gy$ | dM, BA |
| $\dashv\vdash$ | $\exists g \forall x \exists y.\ \overline{fxy} \Leftrightarrow gy$ | Gen |
| $\dashv$ | $\forall x \exists y.\ \overline{fxy} \Leftrightarrow (\lambda y.\overline{fyy})y$ | $\exists I$, $g = \lambda y.\overline{fyy}$ |
| $\dashv\vdash$ | $\exists y.\ \overline{fxy} \Leftrightarrow \overline{fyy}$ | Gen, $\beta$ |
| $\dashv$ | $\overline{fxx} \Leftrightarrow \overline{fxx}$ | $\exists I$, $y = x$ |
| $=$ | $1$ | BA |

Justification: $s_1 = s_2 \dashv\vdash s_3 \dashv s_4 \dashv\vdash s_5 \dashv s_6 = 1$

$BQ \vdash s_1 = s_2 \quad s_2 \dashv\vdash^{BQ} s_3 \quad s_4 \vdash^{BQ} s_3 \quad s_4 \dashv\vdash^{BQ} s_5 \quad s_6 \vdash^{BQ} s_5 \quad BQ \vdash s_6 = 1$

Hence $BQ \vdash s_1 = 1$.