Identities

Henkin 1963

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Standard Interpretation of Identities

Set $\mathcal{X}$

$\doteq_{\mathcal{X}} \in \mathcal{X} \to \mathcal{X} \to \mathbb{B}$

$\doteq_{\mathcal{X}} x\, y = (x = y)$

$$\boxed{\doteq_{\mathcal{X}} = \{(x,x) \mid x \in \mathcal{X}\}}$$

Constants Assumed

- Boolean constants: $\mathbb{B}, 0, 1, \neg, \wedge, \vee$
- Quantifier constants: for every type T: $\forall_T, \exists_T$
- **Identity constants**: for every type T:
  $\doteq_T : T \to T \to \mathbb{B}$
- Additional constants: whatever you like

Identity Axioms

Reflexivity
Replacement
Extensionality
$\forall$-, $\exists$-Definition

Ref $x \doteq x = 1$

Rep $x \doteq y \wedge fx = x \doteq y \wedge fy$

∀D $\forall f = f \doteq \lambda x.1$

∃D $\exists f = \overline{f \doteq \lambda x.0}$

$\}$ quantifiers are defined

Ext $\forall x. fx \doteq gx = f \doteq g$

semantically redundant

Identity Axioms

$\mathrm{IA}_{T,T'} \overset{\text{def}}{=} \{\mathrm{Ref}_T, \mathrm{Rep}_T, \mathrm{Ext}_{T,T'}, \forall D_T, \exists D_T\}$

$\mathrm{IA} \overset{\text{def}}{=} \bigcup \{\mathrm{IA}_{T,T'} \mid T, T' \in T_Y\}$

$\mathrm{BI} \overset{\text{def}}{=} \mathrm{HB} \cup \mathrm{IA}$

$\mathcal{D}$ gives standard interpretation to logical constants

$\Longleftrightarrow \quad \mathcal{D} \vDash \mathrm{BI} \quad \wedge \quad \mathcal{D}0 \neq \mathcal{D}1$

up to isomorphism

Remarks

1. Duality is lost since dual of $\doteq$ is not present. Can be fixed by introduction of dual constants $\not\doteq$ and defining axioms $x \not\doteq y = \overline{x \doteq y}$.

2. Implicational versions

   Rep': $x \doteq y \wedge fx \to fy = 1$

   Ext': $(\forall x.\ fx \doteq gx) \to f \doteq g = 1$

   of Rep and Ext are deductively equivalent.

Boolean Identity Agrees (BIA)

$\mathrm{BI} \vdash x \doteq y = x \Leftrightarrow y$ if $x, y : \mathbb{B}$

Proof.

$$
\begin{aligned}
& x \doteq y \\
&= x \doteq y \wedge (\lambda y.\ x \Leftrightarrow y)x && \beta, \mathrm{BA} \\
&= x \doteq y \wedge (\lambda y.\ x \Leftrightarrow y)y && \mathrm{Rep} \\
&= x \doteq y \wedge x \Leftrightarrow y && \beta
\end{aligned}
$$

$$
\begin{aligned}
& x \Leftrightarrow y \\
&= x \Leftrightarrow y \wedge (\lambda y.\ x \doteq y)x && \beta, \mathrm{Ref}, \mathrm{BA} \\
&= x \Leftrightarrow y \wedge (\lambda y.\ x \doteq y)y && \mathrm{BRep} \\
&= x \Leftrightarrow y \wedge x \doteq y && \beta
\end{aligned}
$$

□

Symmetry, Transitivity, Right Congruence

$\mathrm{BI} \vdash x \doteq y = y \doteq x$

$\mathrm{BI} \vdash x \doteq y \wedge y \doteq z \to x \doteq z$

$\mathrm{BI} \vdash x \doteq y \to fx \doteq fy$

follow similar to BIA with Rep, β, GR

$BI \vdash BQ$

To show: $BI \vdash \forall x.1 = 1$, $\forall f = \forall f \wedge fx$, $\exists x.0 = 0$, $\exists f = \exists f \vee fx$

$\forall 1$ $\quad \forall x.1 \overset{\forall D}{=} (\lambda x.1) \doteq (\lambda x.1) \overset{R}{=} 1$

$\forall I$ $\quad \forall f = f \doteq (\lambda x.1) \wedge (\lambda g.\ fx \Leftrightarrow gx)\, f \qquad \forall D, \beta, BA$

$\quad = \forall f \wedge fx \Leftrightarrow (\lambda x.1)x \qquad Rep, \beta, \forall D$

$\quad = \forall f \wedge fx \qquad \beta, BA$

$\exists 0$, $\exists I$ follow with dual arguments

Leibniz

$BI \vdash x \doteq y = \forall f.\ fx \to fy$

Proof. $x \doteq y$

$= x \doteq y \wedge (\lambda y.\ \forall f.\ fx \to fy)\, x \qquad \beta, \forall 1, BA$

$= x \doteq y \wedge (\lambda y.\ \forall f.\ fx \to fy)\, y \qquad Rep$

$= x \doteq y \wedge \forall f.\ fx \to fy \qquad \beta$

$\forall f.\ fx \to fy$

$= (\forall f.\ fx \to fy) \wedge ((\lambda y.\ x \doteq y)x \to (\lambda y.\ x \doteq y)y) \qquad \forall I, f = \lambda y.x \doteq y$

$= (\forall f.\ fx \to fy) \wedge (x \doteq x \to x \doteq y) \qquad \beta$

$= (\forall f.\ fx \to fy) \wedge x \doteq y \qquad Ref, BA \quad \square$

Implicational Proof of Leibniz

Implicational proof is concise than equational proof because it deletes redundant conjuncts with W

$BI \vdash x \doteq y = \forall f.\ fx \to fy$

Proof. $x \doteq y$

$= x \doteq y \wedge (\lambda y.\ \forall f.\ fx \to fy)\, x \qquad \beta, \forall 1, BA$

$\vdash_0 (\lambda y.\ \forall f.\ fx \to fy)\, y \qquad Rep, W$

$= \forall f.\ fx \to fy \qquad \beta$

$\forall f.\ fx \to fy$

$\vdash_0 ((\lambda y.\ x \doteq y)x \to (\lambda y.\ x \doteq y)y) \qquad \forall I\ f = \lambda y.x \doteq y, W$

$= (x \doteq x \to x \doteq y) \qquad \beta$

$= x \doteq y \qquad Ref, BA \quad \square$

Ext'

$BI \vdash \forall x.\ s \doteq t = (\lambda x.s) \doteq (\lambda x.t)$

Proof $\forall x.\ s \doteq t$

$= \forall x.\ (\lambda x.s)x \doteq (\lambda x.t)x \qquad \beta$

$= (\lambda x.s) \doteq (\lambda x.t) \qquad Ext \quad \square$

Internal Replacement with Capture (Rep')

$$\mathrm{BI} \vdash \quad (\forall \bar{x}.\, s \doteq s') \wedge t[s] \rightarrow t[s']$$

if replacement $t[s] \rightsquigarrow t[s']$ captures only variables in $\bar{x}$

t is a blueprint for a term that contains a hole

Proof $(\forall \bar{x}.\, s \doteq s') \wedge t[s]$

$= (\forall \bar{x}.\, s \doteq s') \wedge t[(\lambda \bar{x}.s)\,\bar{x}]$ β

capture-free replacement

$= (\lambda \bar{x}.s) \doteq (\lambda \bar{x}.s') \wedge (\lambda y.\, t[y\,\bar{x}])(\lambda \bar{x}.s)$ Ext', β

$\vdash_0 t[(\lambda \bar{x}.s')\,\bar{x}]$ Rep, β, W

$= t[s']$ β □

Essence of Internal Rep. with Capture

Given $\forall x_1 \ldots x_n.\, s \doteq s'$, we can replace s with s', where capture of $x_1, \ldots, x_n$ is allowed.

Boolean Constants Can be Axiomatized by Definition

$$\mathrm{DBA} \overset{\mathrm{def}}{=} \{ \quad \begin{aligned} 1 &= (\lambda x.x) \doteq (\lambda x.x) \\ 0 &= (\lambda x.x) \doteq (\lambda x.1) \\ \neg x &= x \doteq 0 \\ x \wedge y &= (\lambda f.\, f x y) \doteq (\lambda f.\, f 1 1) \\ x \vee y &= \overline{\bar{x} \wedge \bar{y}} \quad \} \end{aligned}$$

$$\boxed{\mathrm{BI} \vdash\dashv \mathrm{IA} \cup \mathrm{DBA} \cup \{\mathrm{BRep}\}}$$

Proof "$\vdash$" not difficult, exercise.

"$\dashv$" painful, see Andrew's Book. □

Open Problem

$$\boxed{x \doteq y \;\; \vdash^{\mathrm{BI}}_{0} \;\; x = y \quad ?}$$

Does internal identity imply external identity?

Facts:

$\boxed{x \doteq y \;\; \vDash^{\mathrm{BI}}_{0} \;\; x = y}$

$\boxed{x \doteq_{B} y \;\; \vDash^{\mathrm{BI}}_{0} \;\; x = y}$ BIA

$\boxed{x = y \;\; \vdash^{\mathrm{BI}}_{0} \;\; x \doteq y}$ Ref