Equational Deduction

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Language S

Interpretation | Deduction

$\mathcal{D} \models s = t$ | $E \vdash s = t$

Validity | Deducibility

Soundness Properties

1) $\mathcal{D}\sigma t \in \mathcal{D}(\tau t)$ — Interpretation respects types

2) $\left.\begin{array}{l} \mathcal{D} \models E \\ E \vdash s = t \end{array}\right\} \mathcal{D} \models s = t$ — Deduction preserves validity

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Deduction Rules

Sym $\dfrac{s = t}{t = s}$

Subst $\dfrac{s = t}{\theta s = \theta t}$

Congr $\dfrac{s = s'}{st = s't}$ $\quad \dfrac{t = t'}{st = st'}$ $\quad \dfrac{t = t'}{\lambda x.t = \lambda x.t'}$

β $\dfrac{}{(\lambda x.s)t \rightarrow s[x := t]}$

η $\dfrac{}{\lambda x.tx = t}$ x not free in t

Trans $\dfrac{s = s' \quad s' = t}{s = t}$

Refl $\dfrac{}{t = t}$

Subst and ξ exploit implicit universal quantification

Subst $\dfrac{s = t}{\sigma \circ s = \sigma \circ t}$ no capture

ξ $\dfrac{t = t'}{\lambda x.t = \lambda x.t'}$ capture ok

ξ $\dfrac{t = t'}{\lambda x.t = \lambda x.t'}$??

$$\frac{x = 5}{\lambda x.x = \lambda x.5}$$

ok, since validity of $x = 5$ implies that there is no other value for x than 5

$E \vdash s = t$

$s \xrightarrow{\lambda} t$ $\quad$ $s \xleftrightarrow{E} t$

Reduction

terminating
confluent
unique NFs

Rewriting

Reduction

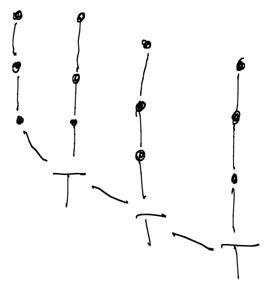
$s \xrightarrow{\lambda} t \overset{def}{\iff} \emptyset \vdash s = t$ with $(\beta|\eta)[Sym]Congr^*$

Rewriting

$s \xleftrightarrow{E} t \overset{def}{\iff} E \vdash s = t$ with $(\beta|\eta|Subst)[Sym]Congr^*$

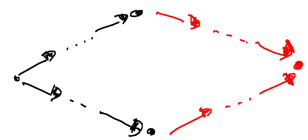
Rewriting Theorem

$$\boxed{E \vdash s = t \iff s = t \ \vee \ s \overset{E^+}{\longleftrightarrow} t}$$



β, η, E Subst
[Sym]
Congr^*

R binary relation

- termination: $\neg\,\exists$ infinite path
- confluence: $\forall$  $\exists$

- normal form for x: terminal node reachable from x
- termination: from every node NF can be reached
- confluence: at most one NF can be reached

Completeness Theorem

The following are equivalent:

1) $\forall \mathcal{D}$: $\quad \mathcal{D} \models s = t$
2) $\emptyset \vdash s = t$
3) s and t have the same $\beta\eta$-NF

Completeness Theorem

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Proof

- (3) $\Rightarrow$ (2) $\Rightarrow$ (1) clear
- (1) $\Rightarrow$ (3) by contraposition
- H. Friedman 1975
  $\exists \mathcal{D}\ \forall s,t$ $\beta\eta$-normal.
  $s \neq t \Rightarrow \mathcal{D} \not\models s = t$

Semantic Entailment Relations

$$\mathcal{D} \models s = t \overset{\text{def}}{\iff} \forall \sigma.\ \mathcal{D}\sigma s = \mathcal{D}\sigma t$$

$$\mathcal{D} \models E \overset{\text{def}}{\iff} \forall e \in E.\ \mathcal{D} \models e$$

$$E \models e \overset{\text{def}}{\iff} \forall \mathcal{D}.\ \mathcal{D} \models E \Rightarrow \mathcal{D} \models e$$

Entailment Relations

$\models$ semantic entailment

$\vdash$ deductive entailment

Soundness Properties

$$\mathcal{D} \models e \Leftarrow \mathcal{D} \models E \wedge E \vdash e$$

$$E \models e \Leftarrow E \vdash e$$

$$E \models E' \Leftarrow E \vdash E'$$

Semantic and Deductive Closure

$$SC\,E \overset{\text{def}}{=} \{e \mid E \models e\}$$

$$DC\,E \overset{\text{def}}{=} \{e \mid E \vdash e\}$$

Soundness: $DC\,E \subseteq SC\,E$

$DC\,E \subseteq SC\,E$ Soundness

$\supseteq$ Completeness

$\boxed{DC\,E \text{ always semi-decidable}}$

$\boxed{\exists E:\ SC\,E \text{ not semi-decidable}}$

Theory of Natural Numbers

with $0, 1, +, \cdot, \leq,$
$\wedge, \vee, \neg,$
$\forall_{\mathbb{N}}, \exists_{\mathbb{N}}$

$Th\,\mathbb{N} \overset{def}{=} \{ e \mid \mathbb{N} \models e \}$

$\boxed{\text{Theo (Gödel 1930)} \quad Th\,\mathbb{N} \text{ not semi-decidable}}$

$\boxed{\text{Theo} \quad \nexists \text{ finite } E : \; SC\,E = Th\,\mathbb{N}}$