Conservative Entailment

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Solutions

σ is solution of E in $\mathcal{D}$ if

$\forall\, s{=}t \in E.\ \ \mathcal{D}\sigma s = \mathcal{D}\sigma t$

$Sol_{\mathcal{D}} E \stackrel{def}{=} \{\sigma \mid \sigma \text{ solution of } E \text{ in } \mathcal{D}\}$

$$\boxed{Sol_{\mathcal{D}}(E_1 \cup E_2) = Sol_{\mathcal{D}} E_1 \cap Sol_{\mathcal{D}} E_2}$$

Solving Equation Systems

Solving E means to derive E' s.t.

1) $Sol_{\mathcal{D}} E = Sol_{\mathcal{D}} E'$
2) Solutions of E' are obvious

E' is called solved form of E

Example

$$\left\{\begin{matrix} x+2y=1 \\ x-y=1 \end{matrix}\right\} \overset{\mathbb{Z}}{\leadsto} \left\{\begin{matrix} x+2y=1 \\ x=y+1 \end{matrix}\right\}$$

$$\leadsto \left\{\begin{matrix} (y+1)+2y=1 \\ x=y+1 \end{matrix}\right\} \overset{\mathbb{Z}}{\leadsto} \left\{\begin{matrix} 3y=0 \\ x=y+1 \end{matrix}\right\}$$

$$\overset{\mathbb{Z}}{\leadsto} \left\{\begin{matrix} x=y+1 \\ y=0 \end{matrix}\right\} \leadsto \left\{\begin{matrix} x=0+1 \\ y=0 \end{matrix}\right\} \overset{\mathbb{Z}}{\leadsto} \left\{\begin{matrix} x=1 \\ y=0 \end{matrix}\right\}$$

Conservative Entailment

$$E \models^{\mathcal{D}}_{\circ} e \overset{df}{\iff} Sol_{\mathcal{D}} E \subseteq Sol_{\mathcal{D}}\{e\}$$

$$\boxed{E \models^{\mathcal{D}}_{\circ} e \implies \varphi\theta E \models^{\mathcal{D}} \varphi\theta e}$$

stable under substitution

Conservative Deduction

$$E \vdash_{\circ} e \overset{df}{\iff} E \vdash e \text{ without Subst and capture}$$

$$E \vdash^{A}_{\circ} e \overset{df}{\iff} E \cup DCA \vdash_{\circ} e$$

$$\boxed{\mathcal{D} \models A \wedge E \vdash^{A}_{\circ} e \implies E \models^{\mathcal{D}}_{\circ} e}$$

Soundness

$$\boxed{E \vdash^{A}_{\circ} e \implies \varphi\theta E \vdash^{A}_{\circ} \varphi\theta e}$$

Stability under substitution

Conservative Rewriting

$$s \overset{E}{\longleftrightarrow}_{\circ} t \overset{df}{\iff} s \overset{E}{\longleftrightarrow} t \text{ without Subst and without capturing}$$

$$\boxed{E \vdash^{A}_{\circ} s = t \iff s = t \vee s \left(\overset{E}{\longleftrightarrow}_{\circ} \cup \overset{A}{\longleftrightarrow}\right)^{+} t}$$

Capture-free Replacement

$$E_1 \dashv\vdash^{E}_{\circ} E_2 \overset{df}{\iff} E_1 \vdash^{E}_{\circ} E_2 \wedge E_2 \vdash^{E}_{\circ} E_1$$

$$\boxed{E \cup \{x = y\} \dashv\vdash_{\circ} E[x := y] \cup \{x = y\}}$$