

Clause Forms (DNF, CNF)

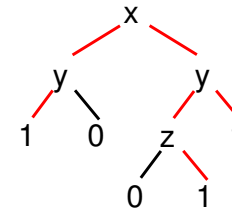
Willard V. Quine.
On Cores and Prime Implicants of Truth Functions.
American Mathematical Monthly, 1959.

G. Smolka

6-7

May 20, 2005

Decision Tree \rightarrow DNF

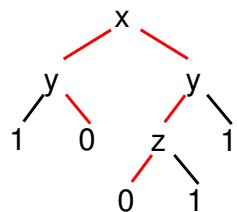


$$(\neg x \wedge \neg y) \vee (x \wedge \neg y \wedge z) \vee (x \wedge y)$$

yields 1 iff one of the clauses yields 1

8

Decision Tree \rightarrow CNF



$$(x \vee \neg y) \wedge (\neg x \vee y \vee z)$$

yields 0 iff one of the clauses yields 0

10

Conjunctive Normal Forms

$$\text{CNF} = 1 \mid DC_1 \wedge \dots \wedge DC_n \quad \text{where } n \geq 1$$

$$DC = 0 \mid L_1 \vee \dots \vee L_n \quad \text{disjunctive clause}$$

where $n \geq 1$ and no variable occurs more than once

$$L = x \mid \neg x \quad \text{literal}$$

$$\forall \text{ B. term } \exists \text{ equiv. CNF}$$

Disjunctive Normal Forms

$$DNF = \bigvee (C_1 \vee \dots \vee C_n) \quad \text{where } n \geq 1$$

$$CC = \emptyset \mid L_1 \wedge \dots \wedge L_n \quad \text{Conjunctive clause}$$

where $n \geq 1$ and no variable occurs more than once

$$L = x \mid \neg x \quad \text{literal}$$

$$\forall \text{ B. term } \exists \text{ equiv. DNF}$$

$$\bigwedge \text{ CNF equiv. to } t \iff \bigwedge \text{ DNF equiv. to } \bar{t}$$

Literal Clause Sets

- clause is called **literal** if it contains only literals
- clause set is called **literal** if it contains only literal clauses

$$S, S' \text{ literal clause sets: } S, S' \text{ conj. equiv.} \iff S, S' \text{ disj. equiv.}$$

Proof: Duality.

equivalent wrt conjunctive interpretation (i.e., S, S' describe same B.-function)

Clause Sets

good representations for CNFs and DNFs

Clause C : finite set of B. terms

Clause set S : finite set of clauses

$$\text{Conjunctive interpretation} \quad \bigwedge_{C \in S} t$$

$$\text{Disjunctive interpretation} \quad \bigvee_{C \in S} t$$

$$\bigwedge_{\emptyset} = \top, \quad \bigvee_{\emptyset} = \perp$$

Normal Clause Sets

- C **trivial** if $\exists t \in C. \bar{t} \in C$
- C **normal** if C literal and not trivial
- S **normal** if all clauses of S are normal

Normal clause sets represent CNFs and DNFs

Explicitness

For every normal clause set S :

$$S \text{ conj. equiv. to } \top \iff S \text{ disj. equiv. to } \perp \iff S = \emptyset$$

CNFs and DNFs are not canonical

$$X = XY + X\bar{Y}$$

$$X = (X+Y)(X+\bar{Y})$$

will define conjunctive and disjunctive

prime forms that are canonical
(set representation required)

Resolvents

$(C - \{t\}) \cup (D - \{\bar{t}\})$ **resolvent for S**

if $C, D \in S$ and $t \in C$ and $\bar{t} \in D$

Addition of resolvents is equivalence transformation w.r.t both interpretations

$$XY + \bar{X}Z = X\bar{Y} + \bar{X}Z + YZ \quad (\text{Resolution})$$

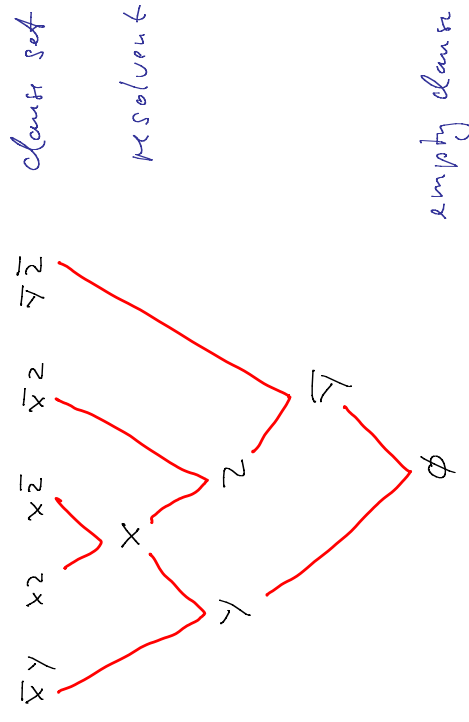
Redundant Clauses

C redundant for S if
either C trivial
or $\exists D \in S, D \subseteq C$

Deletion of redundant clauses is equivalence transformation w.r.t both interpretations

$$\begin{aligned} \bar{X} + \bar{Y}YZ &= X && (\text{Compl, Id}) \\ X + Y + YZ &= X + Y && (\text{Absorption}) \end{aligned}$$

Example



Proof of Termination

Cl_a = set of all clauses containing only terms appearing in initial clause set finite!

$Red\ S = \{ C \in Cl_a \mid C \text{ redundant for } S \}$

- Idea: $Red\ S_1 \subseteq Red\ S_2 \subseteq Red\ S_3 \subseteq \dots \subseteq Cl_a$
- Deletion of a redundant clause doesn't change $Red\ S$
- Addition of a non-redundant resolvent makes $Red\ S$ larger. □

- 1) Addition of resolvents and deletion of redundant clauses are equivalence transformations w.r.t both interpretations
- 2) Adding non-redundant resolvents and deleting redundant clauses terminates

Prime Forms

- A clause set is called **prime form** if it
- is literal
 - contains no redundant clauses
 - has no non-redundant resolvent

Resolution Theorem

\forall prime form $S \ \forall$ literal clause C :
 S equiv. to $S \cup \{C\} \iff C$ redundant for S

Proof in lecture notes 2004

= syntactic redundancy

Consequences of Resolution Theorem

Different prime forms denote different β -functions (w.r.t both interpretations)

Canonicity

For every prime form S :

S conj. equiv. to $0 \iff S$ disj. equiv. to $1 \iff S = \{\emptyset\}$

Explicitness

\forall prime form $S \ \forall$ literal clause set S' :
 S, S' equiv. $\implies \forall C \in S'$: C redundant for S

CPF's and DPF's

A prime form S is called a **CPF** [DPF] for a term t if S is equivalent to t w.r.t the conjunctive [disjunctive] interpretation.

Proof easy with the previous theorems

- Every term has exactly one CPF and DPF
- For all S, t terms s.t the following are equivalent
 - (1) S and t are equivalent
 - (2) S and t have the same CPF
 - (3) S and t have the same DPF

Example

$E \mapsto \{t = \neg\}$ normalization
 \mapsto CNF for t
 \mapsto CPF for t
 \mapsto solved form for E

