Symmetric Sequent Calculus

Gerhard Gentzen, 1935
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First step towards natural deduction systems

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Sequent: $C \Rightarrow D$ where C, D clauses

- Interpretation: $\bigwedge_{s \in C} s \rightarrow \bigvee_{t \in D} t$
- $C \Rightarrow D$ **valid** if $\mathcal{I} \models C \Rightarrow D = 1$
- Sequent can be seen as disjunctive clause:
- $\underbrace{\{s_1, \dots, s_m\}}_{\wedge} \Rightarrow \underbrace{\{t_1, \dots, t_n\}}_{\vee} \rightsquigarrow \underbrace{\{\overline{s}_1, \dots, \overline{s}_m, t_1, \dots, t_n\}}_{\vee}$
- literal clause $\rightsquigarrow$ sequent that contains no constants

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Deduction Rules for Sequents $\dfrac{\sigma_1 \ \dots \ \sigma_n}{\sigma}$

- A rule $\dfrac{\sigma_1 \ \dots \ \sigma_n}{\sigma}$ is called ($\sigma_1, \dots, \sigma_n, \sigma$ are sequents)
  - **sound** if $\mathcal{I} \models \sigma_1 \wedge \dots \wedge \sigma_n \rightarrow \sigma = 1$
  - **invertible** if $\mathcal{I} \models \sigma_1 \wedge \dots \wedge \sigma_n = \sigma$
- A set of rules is called **complete** if every valid sequent can be deduced with the rules
- **Guiding idea**: Find invertible deduction rules and that backward application of rules computes CNF
(recall: valid CNF is empty)

Rules for Conjunction

$$\frac{C, s, t \Rightarrow D}{C, s \wedge t \Rightarrow D} \qquad \frac{C \Rightarrow D, s \qquad C \Rightarrow D, t}{C \Rightarrow D, s \wedge t} \qquad \downarrow \text{forward} \quad \uparrow \text{backward}$$

$$\boxed{\text{Each constant will be treated by exactly 2 rules}}$$

Forward application

- constructs terms
- introduces constants
- diverges

Backward application

- decomposes terms
- eliminates constants
- does not introduce new terms (subterm property)
- terminates

Complete Set of Invertible Rules

$\top$ $\quad \dfrac{}{C, t \Rightarrow D, t}$ $\quad$ trivial sequents

1 $\quad \dfrac{C \Rightarrow D}{C, 1 \Rightarrow D} \qquad \dfrac{}{C \Rightarrow D, 1}$

0 $\quad \dfrac{}{C, 0 \Rightarrow D} \qquad \dfrac{C \Rightarrow D}{C \Rightarrow D, 0}$

$\neg$ $\quad \dfrac{C \Rightarrow D, s}{C, \neg s \Rightarrow D} \qquad \dfrac{C, s \Rightarrow D}{C \Rightarrow D, \neg s}$

$\wedge$ $\quad \dfrac{C, s, t \Rightarrow D}{C, s \wedge t \Rightarrow D} \qquad \dfrac{C \Rightarrow D, s \quad C \Rightarrow D, t}{C \Rightarrow D, s \wedge t}$

$\vee$ $\quad \dfrac{C, s \Rightarrow D \quad C, t \Rightarrow D}{C, s \vee t \Rightarrow D} \qquad \dfrac{C \Rightarrow D, s, t}{C \Rightarrow D, s \vee t}$

Sequent Proof in Tree Notation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \Rightarrow x, xy}\,\top}{x, \bar{x} \Rightarrow xy}\,\neg L \qquad \dfrac{\dfrac{}{x, y \Rightarrow x}\,\top \quad \dfrac{}{x, y \Rightarrow y}\,\top}{x, y \Rightarrow xy}\,\wedge R}{x, \bar{x}+y \Rightarrow xy}\,\vee L}{x(\bar{x}+y) \Rightarrow xy}\,\wedge L$$

Sequent Proof in Linear Notation

$(x, \bar{x}+y \Rightarrow xy)$

$\dashv \ (x, \bar{x} \Rightarrow xy)(x, y \Rightarrow xy)$ $\qquad \vee L$

$\dashv \ (x, \bar{x} \Rightarrow x)(x, \bar{x} \Rightarrow y)(x, y \Rightarrow x)(x, y \Rightarrow y)$ $\qquad 2 \times \wedge R$

$\dashv \ (x, \bar{x} \Rightarrow y)$ $\qquad 3 \times \top$

$\dashv \ (x \Rightarrow y, x)$ $\qquad \neg L$

$\dashv$ $\qquad \top$

- If a sequent contains a constant, then a rule is backward applicable
- Sequents, to which no rule is backward applicable correspond to normal clauses

Proof of Invertibility

Show that each rule is equivalence transformation

For instance: $\quad \dfrac{C \Rightarrow D, s \quad C \Rightarrow D, t}{C \Rightarrow D, s \wedge t}$

$$(C \rightarrow D+s)(C \rightarrow D+t) = (\bar{C}+D+s)(\bar{C}+D+t)$$

$$= \bar{C}+D+st \qquad \text{Distri}$$

$$= C \rightarrow D+st$$

Completeness Proof

- $\emptyset$ is the only CNF equivalent to 1
- If S is CNF for s and s equiv. to 1, then $S = \emptyset$
- If a sequent s is equiv. to 1, then backward application yields the empty CNF
- Seen forward, this is a derivation $\vdash s$ $\square$

Modular Completeness

To derive a sequent s, at most the rules for the constants occuring in s are needed (besides T)

Some sound but non-invertible rules

Weakening $\quad \dfrac{C \Rightarrow D}{C, s \Rightarrow D} \qquad \dfrac{C \Rightarrow D}{C \Rightarrow D, s}$

Cut $\quad \dfrac{C \Rightarrow s \qquad C, s \Rightarrow D}{C \Rightarrow D}$

Soundness of Cut follows with resolution:

$$(C \to s)(Cs \to D) = (\bar{C} + s)(\overline{Cs} + D)$$
$$= (\bar{C} + s)(\bar{C} + \bar{s} + D)$$
$$= (\bar{C} + s)(\bar{C} + D) \qquad \text{Reso, Absorption}$$
$$= (C \Rightarrow s)(C \Rightarrow D)$$