Natural Deduction (ND)

Gentzen → Curry → Howard
1935 1958 1969 1980

informal formal

- used in proof assistants (e.g., Isabelle)
- proofs are terms (i.e., syntactic objects)
- For informal account of ND see textbook [Huth/Ryan 2000]

Natural sequent: $C \Rightarrow \Gamma$

Find a sound and complete proof system for natural sequents!

- At first, we will only consider the constants $\mathbf{0}$ and $\rightarrow$
- Can describe all (finitary) B. functions

Propositional Types

- Types $t = o \mid X \mid t \rightarrow t$ where B. variables X can appear as constants
- Can be interpreted as B. terms

Proof Terms

Terms $p = \delta_t \mid x \mid pp \mid \lambda x p$ as follows:

- constants δ_t have type $((t \rightarrow o) \rightarrow o) \rightarrow t$ proof by contradiction
- variables x have propositional types

(for every prop. type t there is a constant δ_t)

Theorem (Curry-Howard Correspondence)

For every propositional type t:
t valid $\iff$ t inhabited

$\gamma \models t = \Gamma$ $\exists$ closed proof term p. $\tau p = t$

B. term $\longleftrightarrow$ prop. type

proof $\longleftrightarrow$ proof term

Examples

$X \to Y \to X$

$\lambda xy.\, x$

where $x : X$, $y : Y$

prop. type

proof term

$0 \to X$

$\lambda z.\, \delta(\lambda f.\, z)$

where $z : 0$, $f : X \to 0$

Notation: $X \to Y \to X \;=\; X \to (Y \to X)$

i.e., $\to$ right associative

Example: Contraposition

$\big((X \to 0) \to (Y \to 0)\big) \to Y \to X$

$\lambda f y.\, \delta(\lambda g.\, f\, g\, y)$

where $f : (X \to 0) \to Y \to 0$

$y : Y$

$g : X \to 0$

Challenge Exercise: Peirce's Law

Find a proof term for

$((X \to Y) \to X) \to X$

$\lambda f.\, \delta(\lambda g.\, g\,(f(\lambda x.\, \delta(\lambda h.\, g\, x))))$

$f : (X \to Y) \to X$

$g : X \to 0$

$h : Y \to 0$

proof term for Peirce's Law must use δ

Equational proof is easy:

$((x \to y) \to x) \to x$

$= \overline{\overline{x} + y} + x + x$ — Def $\to$

$= (\overline{x} + y)\,\overline{x} + x$ — de Morg, double Neg

$= \overline{x} + x$ — Absorption

$= 1$ — complement

Curry-Howard Correspondence, General Version

For every natural sequent $C \Rightarrow \Gamma$:

$C \Rightarrow \Gamma$ valid $\iff$

$\exists$ proof term p. $\tau p = \Gamma \;\wedge$

$\forall x \in FV p.\;\; \tau x \in C$

Soundness

$\forall$ proof term p. $\{\tau x \mid x \in FV p\} \Rightarrow \tau p$ valid

Proof by induction on p

Can use all constants whose type is valid

1) types of variables are premises
2) types of constants S_t are valid
3) Validity of applications follows with $\dfrac{C \Rightarrow \sigma \to t \quad C' \Rightarrow \sigma}{C \cup C' \Rightarrow t}$
4) Validity of abstractions follows with $\dfrac{C \Rightarrow \epsilon}{C - \{\sigma\} \Rightarrow \sigma \to \epsilon}$

Sound sequent rules

□

Completeness

Proof translation: symmetric sequent proof $\longrightarrow$ proof term

Proof based on completeness of the following symmetric sequent calculus

T $\quad \dfrac{}{C, t \Rightarrow D, t}$ trivial sequents

O $\quad \dfrac{}{C, 0 \Rightarrow D} \qquad \dfrac{C \Rightarrow D}{C \Rightarrow D, 0}$

$\Rightarrow$ $\quad \dfrac{C \Rightarrow D, \sigma \quad C, t \Rightarrow D}{C, \sigma \to t \Rightarrow D} \qquad \dfrac{C, \sigma \Rightarrow D, t}{C \Rightarrow D, \sigma \to t}$

Lemma (Completeness)

$\vdash C \Rightarrow D \Rightarrow \exists$ proof term p : $C \Rightarrow D \vdash p : 0$

$C \Rightarrow D \vdash p : t \quad \overset{\text{def}}{\Longleftrightarrow} \quad \tau p = t \wedge \forall x \in FV p.$

$\tau x \in C \;\vee$

$\exists \sigma \in D.\ \tau x = \sigma \to 0$

$\exists p.\ C, \sigma \Rightarrow D \vdash p : t \iff \exists p.\ C \Rightarrow D \vdash p : \sigma \to t$

$\exists p.\ C \Rightarrow D, \sigma \vdash p : t \iff \exists p.\ C \Rightarrow D \vdash p : (\sigma \to 0) \to t$

Proof of Completeness Lemma

By induction on Sequent rules

T $\quad \dfrac{}{C, t \Rightarrow D, t}$ $\quad x : 0$

$x : t \qquad f : t \to 0$

OR $\quad \dfrac{C \Rightarrow D}{C \Rightarrow D, 0}$ — $p : 0$

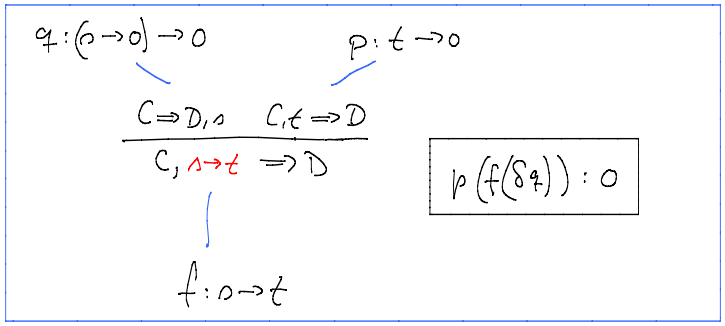
OL $\quad \dfrac{}{C, 0 \Rightarrow D}$ $\quad x : 0$

x

$\to R$ $\quad \dfrac{C, \sigma \Rightarrow D, t}{C \Rightarrow D, \sigma \to t}$ — $p : \sigma \to (t \to 0) \to 0$

$f : (\sigma \to t) \to 0$

$f(\lambda x{:}\sigma.\ \delta(p x)) : 0$



Proof of Completeness

$$C \Rightarrow t \text{ valid} \Rightarrow \exists p.\ \tau p = t \wedge \{\tau x / x \in FV p\} \subseteq C$$

Proof

$C \Rightarrow t$ valid

$\vdash C \Rightarrow t$ completeness of sequent calculus

$\exists p.\ C \Rightarrow t \vdash p : o$ completeness lemma

$\exists p.\ C \Rightarrow \phi \vdash p : (t \to o) \to o$

$\exists p.\ \tau(\delta p) = t \wedge \{\tau x / FV(\delta p)\} \subseteq C$ □

ND without Proof Terms $\vdash^N$

Proof system for sequents $C \Rightarrow o$
(only the constants $\to, o$ are allowed)

T trivial
W weakening
I introduction
MP modus ponens

$$T\ \frac{}{o \Rightarrow o} \qquad W\ \frac{C \Rightarrow t}{C, o \Rightarrow t}$$

$$I\ \frac{C, o \Rightarrow t}{C \Rightarrow o \to t} \qquad MP\ \frac{C \Rightarrow o \to t \quad C' \Rightarrow o}{C \cup C' \Rightarrow t}$$

$$\delta\ \frac{C, o \to o \Rightarrow o}{C \Rightarrow o}$$

$$\vdash^N C \Rightarrow o \iff C \Rightarrow o \text{ valid}$$

Proof "$\Rightarrow$" Ind on $\vdash^N$, easy

"$\Leftarrow$" Lemma, Completeness of proof terms, and Rule W □

Lemma

$$\forall \text{ proof term } p.\ \vdash^N \{\tau x / x \in FV p\} \Rightarrow \tau p$$

Proof. Induction on p, straightforward
except for $p = \delta$

Proof of Lemma continued

$$\dfrac{\dfrac{}{(\Gamma\to\sigma)\to\sigma \Rightarrow (\rho\to\sigma)\to\sigma}\,T \qquad \dfrac{}{\rho\to\sigma\Rightarrow\rho\to\sigma}\,T}{(\Gamma\to\sigma)\to\sigma,\ \rho\to\sigma \Rightarrow \sigma}\,MP$$

$$\dfrac{(\Gamma\to\sigma)\to\sigma,\ \rho\to\sigma \Rightarrow \sigma}{(\Gamma\to\sigma)\to\sigma \Rightarrow \rho}\,S$$

$$\dfrac{(\Gamma\to\sigma)\to\sigma \Rightarrow \rho}{\emptyset \Rightarrow ((\rho\to\sigma)\to\sigma)\to\rho}\,I$$

□

$$I\ \dfrac{C,\rho\Rightarrow\tau}{C\Rightarrow\rho\to\tau} \qquad MP\ \dfrac{C\Rightarrow\rho\to\tau \quad C'\Rightarrow\rho}{C\cup C'\Rightarrow\tau}$$

$$S\ \dfrac{C,\rho\to\sigma\Rightarrow\sigma}{C\Rightarrow\rho}$$

Properties of $\vdash^N$

$$C,\rho\Rightarrow\tau \ \vdash^N\ C\Rightarrow\rho\to\tau$$

$$(\forall C.\ C\Rightarrow\rho \vdash^N C\Rightarrow\tau) \iff \vdash^N \rho\Rightarrow\tau$$

Additional ND-inference rules can be modelled as constants for proof terms (e.g., S)

Proof terms are compact formal representations of ND-proofs

Example: ND-proof of K

$$\dfrac{\dfrac{\dfrac{}{\rho\Rightarrow\rho}\,T}{\rho,\tau\Rightarrow\rho}\,W}{\rho\Rightarrow\tau\to\rho}\,I}{\emptyset\Rightarrow\rho\to\tau\to\rho}\,I$$

$$\lambda x y.\ x$$
$$x:\rho$$
$$y:\tau$$

Example: ND-proof of S

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho\to\tau\to\sigma\Rightarrow\rho\to\tau\to\sigma}\,T \quad \dfrac{}{\rho\Rightarrow\rho}\,T}{\rho\to\tau\to\sigma,\ \rho\Rightarrow\tau\to\sigma}\,MP \quad \dfrac{\dfrac{}{\rho\to\tau\Rightarrow\rho\to\tau}\,T \quad \dfrac{}{\rho\Rightarrow\rho}\,T}{\rho\to\tau,\rho\Rightarrow\tau}\,MP}{\rho\to\tau\to\sigma,\rho,\rho\to\tau\Rightarrow\sigma}\,MP}{\rho\to\tau\to\sigma,\rho,\rho\to\tau\Rightarrow\sigma}\,I}{\rho\to\tau\to\sigma,\rho\to\tau\Rightarrow\rho\to\sigma}\,I}{\rho\to\tau\to\sigma\Rightarrow(\rho\to\tau)\to\rho\to\sigma}\,I}{\emptyset\Rightarrow(\rho\to\tau\to\sigma)\to(\rho\to\tau)\to\rho\to\sigma}\,I$$

$$\lambda f g x.\ f x (g x)$$
$$f:\rho\to\tau\to\sigma$$
$$g:\rho\to\tau$$
$$x:\rho$$

Combinatorial Proof Terms

- If constants S, K are available, all λ's can be eliminated
- Combinatorial proof term $p = K_{\sigma,\tau} \mid S_{\sigma,\tau,\tau'} \mid \delta_\tau \mid pp$

  $K_{\sigma,\tau} : \sigma \to \tau \to \sigma$

  $S_{\sigma,\tau,\tau'} : (\sigma \to \tau \to \tau') \to (\sigma \to \tau) \to \sigma \to \tau'$

For all prop. types τ:
τ valid $\iff$ $\exists$ closed comb. proof term for τ

Combinatorial Proof System

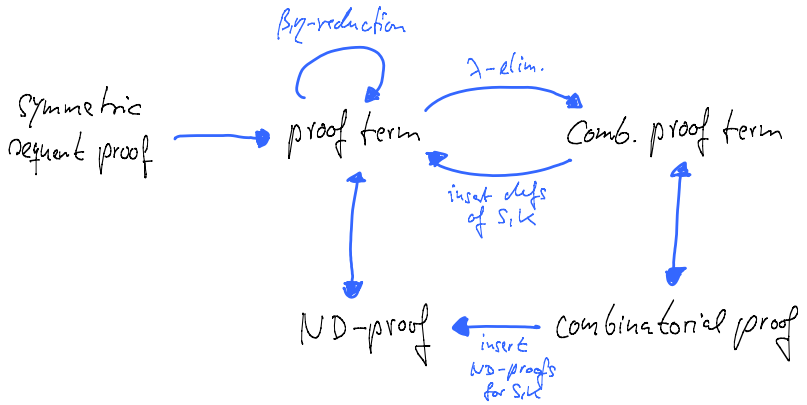
$$\frac{}{\sigma \to \tau \to \sigma}\ K \qquad \frac{}{((\tau \to o) \to o) \to \tau}\ S$$

$$\frac{}{(\sigma \to \tau \to \tau') \to (\sigma \to \tau) \to \sigma \to \tau'}\ S \qquad \frac{\sigma \to \tau \quad \sigma}{\tau}\ MP$$

For all prop. types τ: τ valid $\iff \vdash \tau$

- Combinatorial proof systems where first used by Frege 1879
- First completeness proof by Post 1921
- Combinatorial proof systems are often called Hilbert Systems

Proof Translations



ND rules for $\wedge$ and $\vee$

$$\wedge I\ \frac{C \Rightarrow \sigma \quad C' \Rightarrow \sigma'}{C \cup C' \Rightarrow \sigma \wedge \sigma'} \qquad \wedge EL\ \frac{C \Rightarrow \sigma \wedge \sigma'}{C \Rightarrow \sigma} \qquad \wedge ER\ \frac{C \Rightarrow \sigma \wedge \sigma'}{C \Rightarrow \sigma'}$$

$$\vee IL\ \frac{C \Rightarrow \sigma}{C \Rightarrow \sigma \vee \sigma'} \qquad \vee IR\ \frac{C \Rightarrow \sigma'}{C \Rightarrow \sigma \vee \sigma'} \qquad \vee E\ \frac{C \Rightarrow \sigma \vee \sigma' \quad C, \sigma \Rightarrow \tau \quad C, \sigma' \Rightarrow \tau}{C \Rightarrow \tau}$$

- Soundness and completeness holds
- negation $\neg\sigma$ is handled as $\sigma \to o$
- types with $\wedge, \vee$ are possible and useful for progr. languages
- Note: constants appear only right of $\Rightarrow$

I: introduction, E: elimination, L: left, R: right