Higher-Order Boolean Logic

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Syntax and Standard Interpretation

- Same constants as for first-order B. Logic
  $0, 1 : \mathbb{B}$; $\neg : \mathbb{B} \to \mathbb{B}$; $\wedge, \vee : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$
- no restrictions on variables, terms, equations; e.g., we have the equation $f0 \wedge f1 \to fx = 1$ where f, x are variables
- Same standard interpretation as first-order B. Logic: $\mathcal{I}$
- **$\mathcal{I} \models s = t$ decidable** since all types denote finite sets (e.g., $\mathbb{B}$, $\mathbb{B} \to \mathbb{B}$, $(\mathbb{B} \to \mathbb{B}) \to \mathbb{B}, \dots$)

BA Semantically Incomplete

Not visible with 1st-order equations

- $\exists s = t.\ \mathcal{I} \models s = t \not\Rightarrow BA \models s = t$

  $f0 \wedge f1 \to fx = 1$ invalid in many-valued B. algebras

- Deduction Theorem fails:

  $\exists s = t.\quad s = 1 \models^{BA}_{0} t = 1 \not\Rightarrow BA \models s \to t = 1$

  $s = x \wedge fx,\ t = f1$;

  $\emptyset \not\models x \wedge fx \to f1 = 1$ if $f1 = 0$ and $fb = b$ for 3rd value b

Additional Axiom Regains Semantic Completeness

Boolean Replacement

(BRep) $\boxed{x \leftrightarrow y \wedge fx = x \leftrightarrow y \wedge fy}$

Operator precedence: $\leftrightarrow, \wedge, \vee, \to$

$$HB \overset{\text{def}}{=} BA \cup \{BRep\}$$

$$\boxed{\mathcal{I} \models HB}$$

Duality Theorems Remain Valid

$\boxed{HB \vdash \widehat{BRep}}$ $\rightsquigarrow$ $\boxed{\begin{array}{l} HB \models e \iff HB \models \hat{e} \\ HB \vdash e \iff HB \vdash \hat{e} \end{array}}$

Proof. Since $BA \vdash \widehat{x \leftrightarrow y} = \overline{x \leftrightarrow y}$ it suffices to show:

$$HB \vdash \overline{x \leftrightarrow y} + fx = \overline{x \leftrightarrow y} + fy$$

$$\overline{x \leftrightarrow y} + fx = \overline{x \leftrightarrow y \wedge \overline{fx}} = \overline{x \leftrightarrow y \wedge (\lambda x.\overline{fx})x} \qquad BA, \beta$$

$$= \overline{x \leftrightarrow y \wedge (\lambda x.\overline{fx})y} \qquad BRep$$

$$= \overline{x \leftrightarrow y} + fy \qquad BA, \beta$$

□

Expansion

(Exp) $\boxed{HB \vdash fx = \bar{x} \wedge f0 + x \wedge f1}$ [4]

Follows from

$\boxed{HB \vdash x \wedge fx = x \wedge f1}$ [2]

$x = x \Leftrightarrow 1$

$\boxed{HB \vdash \bar{x} \wedge fx = \bar{x} \wedge f0}$ [3]

$\bar{x} = x \Leftrightarrow 0$

Boolean Instantiation

(BI) $\boxed{HB \vdash f0 \wedge f1 \to fx = 1}$ [5] follows with Expansion, Resolution and Absorption

$\boxed{\mathcal{D} \models HB \implies \mathcal{DB} = \{\mathcal{D}0, \mathcal{D}1\}}$ [6]

Semantic Completeness

$\boxed{\forall e.\ \mathcal{J} \models e \iff HB \models e}$ [7]

follows from the above and 0-1 Theorem

Conjecture: HB Deductively Incomplete

$$\exists e.\ \mathcal{J} \models e \wedge HB \nvdash e$$

Candidate: $f(f(fx)) = fx$

Boolean Case Analysis (BCA)

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$$\boxed{HB \subseteq A: \quad A \vdash e \iff A \vdash e[x:=0] \wedge A \vdash e[x:=1]}$$

Proof. "$\Rightarrow$" by substitution rule

"$\Leftarrow$" Let $e = (s = t)$ and ...

Then $A \vdash (s \Leftrightarrow t)[x:=0] = 1$

$\wedge$ $A \vdash (s \Leftrightarrow t)[x:=1] = 1$

Hence $A \vdash s \Leftrightarrow t = 1$ by (5) with $f := \lambda x.\, s \Leftrightarrow t$

Hence $A \vdash e$ □

Deduction Theorem ($\rightarrow$ Agree)

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$$\boxed{HB \subseteq A: \quad s = 1 \vdash^{A}_{0} t = 1 \iff A \vdash s \rightarrow t = 1}$$

Proof "$\Leftarrow$" easy, MP.

"$\Rightarrow$" Let $s = 1 \vdash^{A}_{0} t = 1$.

$t \left(\xleftrightarrow{A} \cup \xleftrightarrow{s=1}_{0}\right)^{*} 1$ Rewriting Theorem

$(s \rightarrow t) \xleftrightarrow{A}{}^{*} (s \rightarrow 1)$ Lemma next page

$A \vdash s \rightarrow t = s \rightarrow 1$ Rewriting Theorem

$A \vdash s \rightarrow t = 1$ $BA \vdash s \rightarrow 1 = 1$ □

Lemma

needed for Ded Theo

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$$\boxed{\begin{array}{ll} (1) & t_1 \xleftrightarrow{A} t_2 \Rightarrow (s \rightarrow t_1) \xleftrightarrow{A} (s \rightarrow t_2) \\ (2) & t_1 \xleftrightarrow{s=1} t_2 \Rightarrow (s \rightarrow t_1) \xleftrightarrow{HB}{}^{*} (s \rightarrow t_2) \end{array}}$$

Proof. (1) is obvious.

(2). Let $t_1 \xleftrightarrow{s=1} t_2$. Then $\exists\, t, x$ such that

$t_1 = t[x:=s]$ and $t_2 = t[x:=1]$ (wlog)

Hence $s \rightarrow t_1 = \overline{s \wedge \overline{t_1}} = \overline{s \Leftrightarrow 1 \wedge (\lambda x.\, \overline{t})s}$ BA

$= \overline{s \Leftrightarrow 1 \wedge (\lambda x.\, \overline{t})1}$ BRep

$= s \rightarrow t_2$ BA

Hence $(s \rightarrow t_1) \xleftrightarrow{HB}{}^{*} (s \rightarrow t_2)$ Rewriting Theorem □

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$$\boxed{HB \subseteq A: \quad A \vdash s = t \iff s = 1 \dashv\vdash^{A}_{0} t = 1}$$

Proof "$\Rightarrow$": obvious.

"$\Leftarrow$": Let $s = 1 \dashv\vdash^{A}_{0} t = 1$

Then $A \vdash s \rightarrow t = 1$ and $A \vdash t \rightarrow s = 1$ Ded Theo

Hence $A \vdash s \Leftrightarrow t = 1$ BA

Hence $A \vdash s = t$ BA □

Ded Theo is equivalent to BRep

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$BA \subseteq A$: $\left(\forall s = t.\ s = 1 \vdash^{A}_{0} t = 1 \Rightarrow A \vdash s \to t = 1\right)$

$\iff A \vdash BRep$

Proof. "$\Leftarrow$": Already shown.

"$\Rightarrow$": By assumption and (11) it suffices to show:

$$\begin{array}{ccc} x \Leftrightarrow y \wedge fx = 1 & \vdash^{A}_{0} & x \Leftrightarrow y \wedge fy = 1 \\ \mathrm{I}^{A}_{0} & & \mathrm{I}^{A}_{0} \\ x = y,\ fx = 1 & & x = y,\ fy = 1 \end{array}$$

$\square$