Quantifiers

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Standard Interpretation

Set $\mathcal{X}$

$\forall_{\mathcal{X}} \in (\mathcal{X} \to \mathbb{B}) \to \mathbb{B}$

$\forall_{\mathcal{X}} f = (f = \lambda x.1)$

universal quantifier

$\exists_{\mathcal{X}} \in (\mathcal{X} \to \mathbb{B}) \to \mathbb{B}$

$\exists_{\mathcal{X}} f = (f \neq \lambda x.0)$

existential quantifier

[Church 1940]

Constants Assumed

- Boolean constants: $\mathbb{B}, 0, 1, \neg, \wedge, \vee$
- **Quantifier constants:** for every type T:
  - $\forall_T : (T \to \mathbb{B}) \to \mathbb{B}$
  - $\exists_T : (T \to \mathbb{B}) \to \mathbb{B}$
- Additional constants: whatever you like

Notation

$$\forall x.s \overset{\text{def}}{=} \forall(\lambda x.s)$$

$$\exists x.s \overset{\text{def}}{=} \exists(\lambda x.s)$$

Quantifier Axioms

$\forall 1$	$\forall x.1 = 1$	$\exists x.0 = 0$	$\exists 0$
$\forall I$	$\forall f = \forall f \cdot fx$	$\exists f = \exists f + fx$	$\exists I$

I: Instantiation

Remark: $\exists 0, \exists I$ can be replaced by $\exists f = \overline{\forall x. \overline{fx}}$

Quantifier Axioms

$QA_T \overset{def}{=} \{\forall 1_T, \forall I_T, \exists 1_T, \exists I_T\}$

$QA \overset{def}{=} \bigcup \{QA_T \mid T \in Ty\}$

$BQ \overset{def}{=} HB \cup QA$

$\boxed{\begin{array}{l}\mathcal{D} \text{ gives standard interpretation to logical constants} \\ \iff \mathcal{D} \models BQ \ \wedge \ \mathcal{D}0 \neq \mathcal{D}1\end{array}}$ up to isomorphism

Logical constants: Boolean constants + quantifier constants

Duality

$\widehat{\forall_T} = \exists_T \qquad \widehat{\exists_T} = \forall_T$

$\boxed{\widehat{QA_T} = QA_T}$

$\boxed{BQ \vdash e \iff BQ \vdash \hat{e}}$

$\boxed{BQ \models e \iff BQ \models \hat{e}}$

Duality theorems

Quantifier Laws

Deductive consequences of BQ

dM	$\overline{\forall x. fx} = \exists x. \overline{fx}$
$\forall\wedge$	$\forall f \cdot q = \forall x. fx \cdot q$
$\forall\vee$	$\forall f + q = \forall x. fx + q$
$\forall\to$	$q \to \forall f = \forall x. q \to fx$
$\exists\to$	$\exists f \to q = \forall x. fx \to q$
$\forall E$	$\forall x. q = q$
$\forall\wedge'$	$\forall f \cdot \forall g = \forall x. fx \cdot gx$
$\forall\forall$	$\forall x \forall y. hxy = \forall y \forall x. hxy$
$\forall B$	$\forall k = k0 \cdot k1$

$\boxed{\begin{array}{l} x : T \\ f, g : T \to B \\ q : B \\ h : T \to T' \to B \\ k : B \to B \end{array}}$

E: Elimination

dM: de Morgan

B: Bool

To deduce the quantifier laws from BQ, we will use the following facts from Boolean Logic:

ID $\quad x=y \vdash^{BA}_{0} x\to y=1,\ y\to x=1$ — Implicational Deduction

UoC $\quad x=\overline{y} \vdash^{BA}_{0} x\cdot y=0,\ x+y=1$ — Uniqueness of Complements

BI $\quad HB\vdash f_0\cdot f_1 = f_0\cdot f_1\cdot f_x$ — Boolean Instantiation

BCA $\quad HB\subseteq A:\ A\vdash e \iff A\vdash e[x:=0] \wedge A\vdash e[x:=1]$ — Boolean Case Analysis

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$\forall\wedge$, $\forall\vee$, $\forall\!\to$, $\forall E$ follow by BCA

Claim. $BQ\vdash \forall f\cdot q = \forall x.\, fx\cdot q$

Proof. By BCA.

q=0. $\forall f\cdot q = \forall f\cdot 0 \overset{BA}{=} 0$

$\forall x.\, fx\cdot q = \forall x.\, fx\cdot 0 \overset{BA}{=} \forall x.0 \overset{\forall I}{=} (\forall x.0)\cdot 0 \overset{BA}{=} 0$

q=1. $\forall f\cdot q = \forall f\cdot 1 \overset{BA}{=} \forall f$

$\forall x.\, fx\cdot q = \forall x.\, fx\cdot 1 \overset{BA}{=} \forall x.\, fx \overset{\eta}{=} \forall f$ $\square$

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dM follows by UoC

Claim. $BQ\vdash \overline{\forall x. fx} = \exists x.\overline{fx}$

Proof. By UoC.

1) $(\forall x. fx)(\exists x.\overline{fx}) \overset{\exists 1}{=} \exists x.\,(\forall x. fx)\cdot\overline{fx}$

$\overset{\forall I}{=} \exists x.\,(\forall x. fx)\cdot fx\cdot\overline{fx} \overset{BA}{=} \exists x.0 \overset{\exists 0}{=} 0$

2) $(\forall x. fx)+(\exists x.\overline{fx}) \overset{\forall\vee}{=} \forall x.\, fx + \exists x.\overline{fx}$

$\overset{\exists I}{=} \forall x.\, fx + (\exists x.\overline{fx}) + \overline{fx} \overset{BA}{=} \forall x.1 \overset{\forall 1}{=} 1$ $\square$

$\exists\!\to$ follows with $\widehat{dM}$ and $\forall\vee$

$\forall\land'$ and $\forall\forall$ follow by ID

Claim. $BQ \vdash \forall f \land \forall g = \forall x.\ fx \land gx$

Proof. By ID.

1) $\forall f \land \forall g \to \forall x.\ fx \land gx$

$\overset{\forall\to}{=}\ \forall x.\ \forall f \land \forall g \to fx \land gx$

$\overset{\forall I}{=}\ \forall x.\ \forall f \land fx \land \forall g \land gx \to fx \land gx$

$\overset{BA}{=}\ \forall x.\ 1$

$\overset{\forall 1}{=}\ 1$

2) $(\forall x.\ fx \land gx) \to \forall f \land \forall g$

$\overset{\forall\land,\forall\to}{=}\ \forall x \forall y.\ \underbrace{(\forall x.\ fx \land gx)}_{\wedge} \to fx \land gy$

$\overset{\forall I}{=}\ \forall x \forall y.\ 1 \land fx \land gx \land fy \land gy \to fx \land gy$

$\overset{BA}{=}\ \forall x.\ 1$

$\overset{\forall 1}{=}\ 1$ ☐

Proof of $\forall B$ is Easy

Claim. $BQ \vdash \forall\ell = \ell 0 \land \ell 1$

Proof.

$\forall\ell = \forall\ell \land \ell 0 \land \ell 1$	$\forall I$
$= (\forall q.\ \ell q) \land \ell 0 \land \ell 1$	η
$= \forall q.\ \ell q \land \ell 0 \land \ell 1$	$\forall\land$
$= \forall q.\ \ell 0 \land \ell 1$	BI, BA
$= \ell 0 \land \ell 1$	$\forall E$ ☐

$\forall$Agree

$$\boxed{\forall x.s = 1 \ \overset{BQ}{\vdash\!\dashv}\ s = 1}$$

$E \vdash^{A} e \ \overset{\text{def}}{\iff}\ E \cup A \vdash e$

Proof.

"$\vdash$":

$s = \forall x.s$	Assumption
$= (\forall x.s) \land (\lambda x.s)x$	$\forall I$
$= 1 \land s$	Assumption, β
$= s$	BA

"$\dashv$":

$\forall x.s = \forall x.1$	Assumption, Capture!
$= 1$	$\forall 1$ ☐

Example: Turing's Law

There is no barber who shaves everyone who doesn't shave himself

Reason for undecidability of Halting Problem

$$\boxed{BQ \vdash \ \forall f.\ \overline{\exists x \forall y.\ fxy \leftrightarrow \overline{fyy}} = 1}$$

$f : T \to T \to B$

Proof.

$\forall f.\ \overline{\exists x \forall y.\ fxy \leftrightarrow \overline{fyy}}$	
$= \forall f\ \forall x\ \overline{\exists y.\ fxy \leftrightarrow \overline{fyy}}$	dM
$= \forall f\ \forall x\ \underbrace{\exists y.\ fxy \leftrightarrow fyy}_{\wedge}$	BA
$= \forall f \forall x.\ s + fxx \leftrightarrow fxx$	$\exists I,\ y{:=}x$
$= \forall f \forall x.\ 1$	BA
$= 1$	$\forall 1$ ☐

Turing's Theorem

There is no TM x such that
for every TM y the following holds:
x halts on rep of y
$\iff$ y does not hold on rep of y

TM: Turing Machine
rep: representation

Example: Cantor's Law

Cantor's Diagonalization Argument

Reason why $\mathcal{P}X$ is larger than X

$$\boxed{BQ \vdash \overline{\exists f \forall g \exists x \forall y.\ fxy \Leftrightarrow gy} = 1}$$

$f : T \to T \to B$

Proof.

$$\begin{aligned}
& \overline{\exists f \forall g \exists x \forall y.\ fxy \Leftrightarrow gy} && \\
&= \forall f \underbrace{\exists g \forall x \exists y.\ fxy \Leftrightarrow \overline{gy}}_{s} && dM,\ BA \\
&= \forall f.\ s + \forall x \exists y.\ fxy \Leftrightarrow \overline{(\lambda y.\ \overline{fyy})\, y} && \exists I,\ g := \lambda y.\ \overline{fyy} \\
&= \forall f \forall x.\ s + \underbrace{\exists y.\ fxy \Leftrightarrow fyy}_{t} && \forall v,\ \beta,\ BA \\
&= \forall f \forall x.\ s + t + fxx \Leftrightarrow fxx && \exists I,\ y := x \\
&= \forall f \forall x.\ 1 && BA \\
&= 1 && \forall 1 \quad \square
\end{aligned}$$

Cantor's Theorem

Let X be a set.
Then there exists no surjective function $X \to \mathcal{P}X$. power set of X

$X \rightsquigarrow$ type T
$\mathcal{P}X \rightsquigarrow$ type $T \to B$

$\neg \exists f \forall g \exists x.\ fx \doteq g$
$\forall y.\ fxy \doteq gy$
$\leftrightarrow$

Prenex Forms

$Q_1 x_1 \cdots Q_n x_n.\ s$ where s does not contain a quantifier and every Q_i is a quantifier

s **prenex form of** t if $BQ \vdash s = t$ and s is a prenex form

Prenex forms can be computed with dM, $\forall\wedge$, $\forall\vee$, $\exists\wedge$, $\exists\vee$.
Number of quantifiers can be reduced with $\forall\wedge'$, $\exists\vee'$!

Skolem Forms

$\exists x_1 \dots \exists x_m \forall y_1 \dots y_n . s$ where s contains no quantifier

Skolem forms can be computed with **Skolem Axiom**:

(Sko) $\forall x \exists y.\ hxy = \exists f' \forall x.\ hx(f'x)$

where $x:T,\ y:T'$

Conjecture: $BQ \nvdash Sko$

$$\boxed{BQ,\ Sko \vdash \overline{Sko}}$$

Proof. $\exists x \forall y.\ hxy$

$= \overline{\forall x \exists y.\ (\lambda xy.\overline{hxy})\, xy}$ $\quad dM, \beta, BA$

$= \overline{\exists f' \forall x.\ (\lambda xy.\overline{hxy})\, x\,(f'x)}$ $\quad Sko$

$= \forall f' \exists x.\ hx(f'x)$ $\quad dM, \beta, BA$ $\quad \square$