Boolean Algebra

BA axiomatizes

1) logical operations $\wedge, \vee, \neg$ (Boole 1854)
2) set operations $\cap, \cup$, complement

- first instance of abstract algebra

Tautology Theorem

$\boxed{\begin{array}{l} \text{For every proper model } \mathcal{L} \text{ of BA and} \\ \text{for every B. equation } e \\ BA \vdash e \underset{1}{\iff} BA \models e \underset{2}{\iff} \mathcal{L} \models e \end{array}}$

- Equivalence 1 may be obtained from a completeness theorem for algebraic specifications (Birkhoff 1930)
- Equivalence 2 is a special property of BA
  Properties of logical operations $\hat{=}$ properties of set operations

Quiz

Find an equation e s.t.

1) $\mathcal{J} \models e$
2) $BA \not\models e$

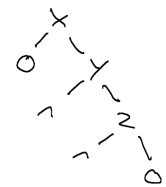
Prime Trees

$\boxed{\forall s \in BT \ \exists^1 t \in PT : \ BA \vdash s = t}$

$\boxed{\begin{array}{l} \text{For all B. terms } s, t \\ BA \vdash s = t \iff \pi s = \pi t \end{array}}$

$PT \subset DT \subset BT$

ordered, reduced



Conditionals

$(s, t_0, t_1) \rightsquigarrow \bar{s} \cdot t_0 + s \cdot t_1$

if $s = 0$ then t_0 else t_1

Surprise: properties of 2-valued conditionals are deducible in BA

$(x, y, y) = y$

$(x, y, z) \circ u = (x, y \circ u, z \circ u)$

$(x, y, z) \circ (x, y', z') = (x, y \circ y', z \circ z')$

0-1 Theorem

$$\boxed{s \in BT \text{ closed} \Rightarrow BA \vdash s = 0 \lor BA \vdash s = 1}$$

Expansion Theorem

$$\boxed{s, x \in BT \Rightarrow BA \vdash s = (x, s[x := 0], s[x := 1])}$$

Suffices to show

1) $BA \vdash x \cdot s = x \cdot (s[x := 1])$

2) $BA \vdash \bar{x} \cdot s = \bar{x} \cdot (s[x := 0])$

Proof by induction on s, using the following tautologies

$x \cdot x = x \cdot 1$ $\qquad x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$

$x \cdot \bar{y} = x \cdot \overline{xy}$ $\qquad x \cdot (y + z) = x \cdot y + x \cdot z$

Canonicity Lemma

$\mathcal{L}$ proper model of BA

s, t different prime trees

$\Rightarrow \exists \mathcal{M} : \mathcal{L} \leq \mathcal{M} \land \hat{\mathcal{M}} s \neq \hat{\mathcal{M}} t$

Proof is somewhat tricky

requires notion of

significant variables $SV_{\mathcal{L}}\, s$

Significant Variables

$\mathcal{L}$ proper model of BA

$SV_{\mathcal{L}} s := \{ x \mid \tau x = B \wedge \exists \mathcal{I}: \mathcal{L} \le \mathcal{I} \wedge \exists v \in \mathcal{L} B: \hat{\mathcal{I}} s \neq \hat{\mathcal{I}}_{x,v} s \}$

F0 $\boxed{BA \models s = t \Rightarrow SV_{\mathcal{L}} s = SV_{\mathcal{L}} t}$

F1 $\boxed{SV_{\mathcal{L}} s \subseteq \mathcal{N} s}$ Coincidence

More Facts

$\mathcal{L}$ proper model of BA, $\mathcal{L} \le \mathcal{I}$

F2 $\boxed{\begin{array}{l} \mathcal{I} x = \mathcal{L} 0 \Rightarrow \mathcal{I}(x, s, t) = \mathcal{I} s \\ \mathcal{I} x = \mathcal{L} 1 \Rightarrow \mathcal{I}(x, s, t) = \mathcal{I} t \end{array}}$

F3 $\boxed{\mathcal{I} s_0 \neq \mathcal{I} s_1 \wedge x \notin \mathcal{N} s_0 \cup \mathcal{N} s_1 \Rightarrow x \in SV_{\mathcal{L}}(x, s_0, s_1)}$ F2

F4 $\boxed{\begin{array}{l} \mathcal{I} s_0 \neq \mathcal{I} t_0 \wedge x \notin \mathcal{N} s_0 \cup \mathcal{N} t_0 \\ \quad \Rightarrow \exists \mathcal{I}': \mathcal{L} \le \mathcal{I}' \wedge \hat{\mathcal{I}}'(x, s_0, s_1) \neq \hat{\mathcal{I}}'(x, t_0, t_1) \end{array}}$ $\mathcal{I}' = \mathcal{I}_{x, \mathcal{L}0}$ F2

$\mathcal{L}$ proper model of BA and $s, t \in PT$. Then
1) $x \in \mathcal{N} s \iff x \in SV_{\mathcal{L}} s$
2) $s \neq t \Rightarrow \exists \mathcal{I}: \mathcal{L} \le \mathcal{I} \wedge \hat{\mathcal{I}} s \neq \hat{\mathcal{I}} t$

Proof. By ind on $|s| + |t|$, both claims together

1) "$\Leftarrow$" by Coincidence. "$\Rightarrow$" by case analysis.
   Case $s \in \{0, 1\}$. ✓
   Case $s = (x, s_0, s_1)$. Follows with IH (2) and F3

2) Case $s, t \in \{0, 1\}$. ✓
   Case "The root variable of s doesn't occur in t or vice versa"
   Follows with (1) and F0
   Case "s and t have the same root variable"
   Follows with IH (2) and F4 □

Modelling with Boolean Equations

- Will consider an application
- Switch to ordinary math mode
- Boolean will mean 2-valued, i.e., $\mathbb{B} = \{0, 1\}$

Diet Rules of an Old Gentleman

1) If you don't drink beer,
   always eat fish

2) If you have both beer and fish,
   don't eat ice cream

3) If you don't drink beer or eat ice cream,
   don't have fish

Model the Diet Rules

with 3 Boolean observables

- B: meal includes beer
- F: meal includes fish
- I: meal includes ice cream

and one equation per rule

$$\boxed{\begin{aligned} &\overline{B} \to F = 1 \\ &B \wedge F \to \overline{I} = 1 \\ &\overline{B} \vee I \to \overline{F} = 1 \end{aligned}}$$

1) If you don't drink beer,
   always eat fish

2) If you have both beer and fish,
   don't eat ice cream

3) If you don't drink beer or eat ice cream,
   don't have fish

When the theory pays off

What you have learned so far in this course doesn't help with the modelling but once you have the model it pays off:

1) Are there meals that satisfy the rules?
   Is the equation system solvable?

2) Are there other (simpler) diet rules that license exactly the same meals?
   Are there equivalent equation systems?

Definition of Solutions

$\mathfrak{A}$: structure

$\mathcal{U} \subseteq Var \cup Con$, $\mathcal{U} \cap \Sigma_{\mathfrak{A}} = \emptyset$ Unknows

$Sol_{\mathfrak{A},\mathcal{U}}\, E := \{ \sigma \mid \exists \mathfrak{I} : \mathfrak{A} \leq \mathfrak{I} \wedge \mathfrak{I} \models E \wedge \sigma \leq \mathfrak{I} \wedge Dom\, \sigma = \mathcal{U} \}$

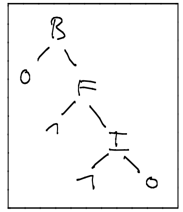
Equation System → Boolean Term

Formalize the equation system E as a B.term s

$$(\bar{B} \to F)\,(B \wedge F \to \bar{I})\,(\bar{B} \vee I \to \bar{F})$$

- Solutions E = Solutions $s = 1$
- E solvable $\iff$ $\mathcal{J} \not\models s = 0$
- $\mathcal{J} \models s = t \implies$ Solutions $s = 1$ = Solutions $t = 1$
- Πs can serve as solved form for E

Solving E ≙ Simplifying s

$\Pi s =$ 

$$\begin{aligned}
s &= (\bar{B} \to F)\,(B \wedge F \to \bar{I})\,(\bar{B} \vee I \to \bar{F}) \\
&\equiv (\bar{\bar{B}} + F)\,(\overline{B \cdot F} + \bar{I})\,(\overline{\bar{B} + I} + \bar{F}) \\
&\equiv (B + F)\,(\bar{B} + \bar{F} + \bar{I})\,(B \cdot \bar{I} + \bar{F}) \\
&\equiv (B + F)\,(\bar{B} + \bar{F} + \bar{I})\,(B + \bar{F})\,(\bar{I} + \bar{F}) \\
&\equiv (\quad)(\quad)(\quad)(\quad)\,B && \text{Resolution} \\
&\equiv (\bar{I} + \bar{F})\,B && \text{Absorption}
\end{aligned}$$