Assignment 6
Introduction to Computational Logic, SS 2007

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Read in the lecture notes: Chapter 4,5

Exercise 6.1 (Eta) Let $sx = tx$ be an equation such that the variable x does not occur in $s = t$.

a) Give a conversion proof of $s = t$ from $\{sx = tx\}$.

b) Give a derivation of $s = t$ from $\{sx = tx\}$. You may use ρ, β, η, Ref, Sym and Trans.

c) Give a derivation of $sx = tx$ from $\{s = t\}$. Use only ρ and β.

Exercise 6.2 (ξ) The ξ-rule

$$\frac{s = t}{\lambda x.s = \lambda x.t} \quad x \in Var$$

requires x to be a variable since otherwise the rule would be unsound.

a) Let a, b be distinct parameters of sort **B**. Argue that $a = b \not\models (\lambda a.a) = (\lambda a.b)$.

b) Let x, y be distinct variables of sort **B**. Argue that $x = y \models (\lambda x.x) = (\lambda x.y)$.

Exercise 6.3 (Contexts) Let C be the context $\lambda s \in Ter.\, Lx(Lys)$. Give the following terms.

a) Cx

b) Cz where $x, y \neq z$

c) $[z := x](\lambda xy.z)$ where $x, y \neq z$

Exercise 6.4 (Contexts) Let x be a variable of type **B**. Find a context C such that $C0 \to C1 \to Cx$ is a formula that is not semantically entailed by PL.
Hint: Choose C such that $\text{PL} \models C0$, $\text{PL} \models C1$, $\text{PL} \models \neg(Cx)$, and Cx captures x. There is a term t such that $\lambda s \in Ter.At(Lxs)$ is a context as required.

Exercise 6.5 (IKS) Let Com be the untyped specification

$$I = \lambda x.x \qquad (I)$$
$$Ky = \lambda x.y \qquad (K)$$
$$Sfg = \lambda x.\, fx(gx) \qquad (S)$$

where I, K, S are parameters and x, y, f, g are variables.

a) For each of the following terms s, find a combinatorial term t and a conversion proof of $s = t$ from *Com*.

   (1) $\lambda xy.y$

   (2) $\lambda xyz.y$

   (3) $\lambda xyz.x$

   (4) $\lambda x.fx(fxx)$

   (5) $(\lambda fx.fx(fx))g$

b) Find a combinatorial specification *Com*′ such that $Com \vDash Com'$.

Exercise 6.6 (BCA) Let s be a formula and x be a variable of type $\mathbf{B}$. Find a derivation of $[x := 0]s \to [x := 1]s \to s$ from PL. You may use ρ, β, η, Ref, Sym, Trans and Subst'.

Exercise 6.7 (I0) Find a derivation of $0 \to x = 1$ from PL. You may use ρ, β, η, Ref, Sym, Trans, Subst, Subst' and Up.

Exercise 6.8 (Generalized BCA) Prove $\mathrm{PL} \vdash f(f(fx)) = fx$ where $x : \mathbf{B}$ and $f : \mathbf{B} \to \mathbf{B}$ are variables. Use the tree notation from the lecture.

Exercise 6.9 (Conversion Proofs) Let *Rep* be the following specification:

Constants	$0, 1 : \mathbf{B}$	
	$\to : \mathbf{B} \to \mathbf{B}$	
	$= : \mathbf{B} \to \mathbf{B} \to \mathbf{B}$	
Axioms	$(x = y) \to fx = (x = y) \to fy$	R1
	$(x = x) = 1$	R2
	$x \to 1 = 1$	R3

For each of the following terms s find a conversion proof of $s = 1$ from *Rep*.

a) $(f1 = 1) \to (f(f(f1)) = f1)$

b) $(f1 = 0) \to (f0 = 1) \to (f(f(f1)) = f1)$

Whenever you use R1, say which term is used for f.

Exercise 6.10 Let *Hoc* be the specification

$(x = y) \to fx = (x = y) \to fy$	BRep
$x = ((y = 0) \to x) \wedge ((y = 1) \to x)$	T1
$(x = x) = 1$	T2
$x \wedge 1 = x$	T3
$x \to 1 = 1$	T4

where $=, 0, 1, \to, \wedge$ are the parameters from PL.

a) Find a conversion proof of $(f(f(f1)) = f1) = 1$ from *Hoc*.

b) Argue that (a) implies $\mathrm{PL} \vdash f(f(f1)) = f1$.