Midterm Exam Introduction to Computational Logic
SS 2007

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Name ______________________________ Seat

Matriculation ______________________________ Code

Please put your ID (or passport) and your student card on your desk.

Open the exam booklet only after you have been asked to do so. After you have opened the exam booklet, it is your obligation to check whether it is complete.

You may only use the exam booklet that carries your name and matriculation. You have to write the exam on the seat with the number that is printed on your exam booklet.

No auxiliary means are allowed. At your desk, you may only have writing utensils, beverages, food, and ID cards. Bags and jackets have to be left at the walls of the lecture room.

If you leave the room without turning in your exam booklet, then this will be judged as an attempt of deception.

If you need to go to the bathroom during the exam, please turn in your exam booklet. Only one person may go to the bathroom at a time.

All solutions have to be written on the right-hand-side pages of the exam booklet. The empty left-hand-side pages may serve as draft paper and **will not be graded**. No other paper is admitted. You may use a pencil.

The exam lasts 150 minutes. You can obtain at most 150 points. The number of points you can get for a problem gives you a hint about how much time you should spend on that problem. For passing the exam it is sufficient to obtain 75 points.

Every attempt of deception will force us to exclude you from this exam and all following exams of this course. The university keeps a record of attempts of deception.

1	2	3	4	5	6	7	8	9	Sum
12	18	25	15	14	16	15	20	15	150

Grade

Problem 1. Axiomatization of Terms (12 points)

Consider the axiomatization of untyped terms. Complete the axioms IL and SL.

IL $\lambda x.s = \lambda y.t \iff$

SL $\theta(\lambda x.t) =$

if

Problem 2. Term Structure (18 points)

Recall de Bruijn's construction of an untyped term structure:

$$\begin{aligned} Nam &:= \mathbb{N} \\ Ter &:= \mathbb{N} \cup (\{1\} \times Ter) \cup (\{2\} \times Ter \times Ter) \\ \lambda s &:= (1, s) \\ st &:= (2, s, t) \end{aligned}$$

Complete the definition of the substitution operator **S** and the abstraction operator L.

$$\mathbf{S}\,\theta\, s \;=\; \mathbf{S}'\, 0\, \theta\, s$$

$$\mathbf{S}'\, d\, \theta\, x \;=$$

$$\mathbf{S}'\, d\, \theta\, (s\, t) \;=$$

$$\mathbf{S}'\, d\, \theta\, (\lambda s) \;=$$

$$\mathit{shift}\, d\, x \;=$$

$$\mathit{shift}\, d\, s \;=$$

$$L\, x\, s \;=$$

Problem 3. Specification of the Natural Numbers (14+3+4+4=25 points)

a) Specify the natural numbers with the constants

$$\begin{aligned} 0, 1 &: \mathbf{B} \\ \rightarrow &: \mathbf{B} \rightarrow \mathbf{B} \rightarrow \mathbf{B} \\ \forall &: (N \rightarrow \mathbf{B}) \rightarrow \mathbf{B} \\ o &: N \\ S &: N \rightarrow N \end{aligned}$$

Do not use other constants but the identities for $\mathbf{B}$ and N.

b) Specify addition $+ : N \rightarrow N \rightarrow N$.

c) Specify a choice function $\mathbf{C} : (N \rightarrow \mathbf{B}) \rightarrow N$.

d) Specify $\exists : (N \rightarrow \mathbf{B}) \rightarrow \mathbf{B}$ with $\mathbf{C}$ and $=$.

Problem 4. Incompleteness (15 points)

Let *Nat* be a specification of the natural numbers that gives the parameters $o : N$ and $S : N \to N$ their canonical meaning and does not contain the parameter $f : N \to N$. Give a set of formulas A such that

i) $Nat \cup A$ has a model

ii) $Nat, A \vDash fx = x$

iii) $Nat, A \nvdash fx = x$

Problem 5. Derivations (7+7=14 points)

Give derivations for the following claims in the basic proof system (ρ, β, η).

a) $sx = tx \ \vdash\ s = t$ where $x \in Var - \mathcal{N}(s = t)$

b) $s = t \ \vdash\ [x := u]s = [x := u]t$ where x is a variable

Problem 6. Combinatorial Specification (16 points)

Give a combinatorial specification of the parameters

$$\begin{aligned} 0,1 &: \mathbf{B} \\ \neg &: \mathbf{B} \to \mathbf{B} \\ \forall,\exists &: (D \to \mathbf{B}) \to \mathbf{B} \\ I &: \mathbf{B} \to \mathbf{B} \\ K &: \mathbf{B} \to \mathbf{B} \to \mathbf{B} \\ K' &: \mathbf{B} \to D \to \mathbf{B} \end{aligned}$$

such that $0, 1, \neg, \forall, \exists$ take their canonical meaning, I is the identity function on $\mathbf{B}$, K satisfies $Kxy = x$, and K' satisfies $K'xz = x$. Do not use other parameters but the identities.

Problem 7. Conversion Proof (15 points)

Let *Con* be the following untyped specification

$$Kx = \lambda y.\,x \qquad\qquad (K)$$

where K is a parameter and x, y are variables. Give a combinatorial term s and a conversion proof of $(\lambda f x y.\, f y)(\lambda x y.\, x) = s$ from *Con*.

Problem 8. Quickies (5·4=20 points)

Answer the following questions. Wrong answers count negative.

a) Give an entailment relation on $\mathbb{Z}$ that is not compact.

b) Give formulas s, t such that $\mathrm{PL}, s \vdash t$ and $\mathrm{PL} \nvdash s \to t$.

c) Is the following statement true? $\mathrm{PL}, x \vee y \vDash x \wedge y$

d) Give a closed specification that is deductively equivalent to $0 \to x = 1$ where $0, 1 \to$ are parameters and x is a variable.

e) Let x, y be two distinct variables of the same type. Is there a term s such that $\lambda x.s = \lambda y.s$?

Problem 9. Boolean Choice (5+10=15 points)

A choice function for a set X is a function $C \in (X \to \mathbb{B}) \to X$ such that $\forall f \in X \to \mathbb{B}\colon\; f(Cf) = \exists_X f$.

a) How many choice functions are there for $\mathbb{B}$?

b) Recall that PL specifies the parameters $0, 1, \to, \neg, \vee, \wedge$ with their usual meaning. Give 2 closed terms that denote different choice functions on $\mathbb{B}$ in the model of PL.