Final Exam Introduction to Computational Logic
SS 2007

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July 21, 2007

Name Seat

Matriculation Code

Please put your ID (or passport) and your student card on your desk.

Open the exam booklet only after you have been asked to do so. After you have opened the exam booklet, it is your obligation to check whether it is complete.

You may only use the exam booklet that carries your name and matriculation. You have to write the exam on the seat with the number that is printed on your exam booklet.

No auxiliary means are allowed. At your desk, you may only have writing utensils, beverages, food, and ID cards. Bags and jackets have to be left at the walls of the lecture room.

If you leave the room without turning in your exam booklet, then this will be judged as an attempt of deception.

If you need to go to the bathroom during the exam, please turn in your exam booklet. Only one person may go to the bathroom at a time.

All solutions have to be written on the right-hand-side pages of the exam booklet. The empty left-hand-side pages may serve as draft paper and **will not be graded**. No other paper is admitted. You may use a pencil.

The exam lasts 150 minutes. You can obtain at most 150 points. The number of points you can get for a problem gives you a hint about how much time you should spend on that problem. For passing the exam it is sufficient to obtain 75 points.

No problem will be graded with less than 0 points.

Every attempt of deception will force us to exclude you from this exam and all following exams of this course. The university keeps a record of attempts of deception.

1	2	3	4	5	6	7	8	9	Sum	**Grade**
14	19	17	18	15	16	16	14	21	150	

Problem 1. Deductive Entailment (7·2=14 points)

Say for each of the following statements whether it is true or false. Wrong answers count negative.

a) $s, t \overset{\mathrm{TL}}{\vdash} s \wedge t$

b) $s \vee t \overset{\mathrm{TL}}{\vdash} s$ or $s \vee t \overset{\mathrm{TL}}{\vdash} t$

c) $A \vdash s = t \implies s \overset{A}{\vDash\!\dashv} t$

d) $s \overset{\mathrm{PL}}{\vdash} t \implies \mathrm{PL} \vdash s \to t$

e) $\mathrm{HL} \vdash st \implies \mathrm{HL} \vdash \exists s$

f) $\overline{\exists s} \overset{\mathrm{HL}}{\vdash\!\dashv} \overline{sx}$ if x is a variable not in $\mathcal{N}s$

g) $\mathrm{PL} \vdash\!\dashv \mathrm{TL} \cup \{(x =_{\mathbf{B}} y) \to (fx =_{\mathbf{B}} fy)\}$ if x, y, f are variables

Problem 2. Case Analysis (19 points)

Let $f : \mathbf{B} \to \mathbf{B}$. Give a tableau proof of $\mathrm{HL} \vdash f(f(f0)) = f0$.

Problem 3. Cantor (17 points)

Give a tableau proof of $\mathrm{HL} \vdash \overline{\forall g \exists x.\, fx = g}$ where $f : T \to T \to \mathbf{B}$.

Problem 4. Double Instantiation (8+10=18 points)

a) Give a tableau proof of HL $\vdash \exists x.\overline{gxy} \vee gzx$.

b) Give a quasi-conversion proof of HL $\vdash (\forall x.\overline{gxy} \wedge gzx) = 0$.

Problem 5. Expansion Rules (5·3=15 points)

We consider tableau proofs from A where $A \vdash \mathrm{HL}$. The soundness conditions for unary and binary expansion rules are as follows:

- $\dfrac{s}{t}$ sound $\iff s \overset{A}{\dashv\vdash} s \vee t$
- $\dfrac{s}{t_1 \mid t_2}$ sound $\iff s \overset{A}{\dashv\vdash} s \vee t_1, s \vee t_2$

Say for each of the following expansion rules whether it is sound or not. Wrong answers count negative.

a) $\dfrac{s \vee t}{s \mid t}$

b) $\dfrac{s \wedge t}{s \mid t}$

c) $\dfrac{s}{t \vee s \mid \bar{t} \vee s}$

d) $\dfrac{(s_1 = s_2) \wedge [x{:=}s_1]t}{[x{:=}s_2]t}$

e) $\dfrac{(s_1 \neq s_2) \vee [x{:=}s_1]t}{[x{:=}s_2]t}$

Problem 6. Conversion Proof (16 points)

Give a conversion proof of $(\forall x \exists y.\, gxy) \to \exists(ga) = 1$ from the axioms

∀I	$\forall f = \forall f \wedge fx$
Imp	$x \wedge y \to y = 1$

where f, x, y are variables. Carry out and annotate every single conversion step. Don't use quasi-conversion.

Problem 7. Conversion Proof with Choice (16 points)

Give a conversion proof of $(\mathrm{C}(\lambda x.x = y) = y) = 1$ from the axioms

Choice	$f(\mathrm{C}f) = \exists f$
∃I	$\exists f = \exists f \vee fx$
Ref	$(x = x) = 1$
Dom	$y \vee 1 = 1$

where f, x, y are variables. Carry out and annotate every single conversion step. Don't use quasi-conversion.

Problem 8. Prime Trees and BDDs (3+3+8=14 points)

Assume the variable order $x < y < z$.

a) Draw the prime tree for $x \to y \to z$.

b) Draw the prime tree for $(x \to y) \to z$.

c) Draw a minimal BDD that has nodes representing the trees from (a) and (b). Mark these nodes with (a) and (b), respectively.

Problem 9. Prime Tree Algorithms (21 points)

Let MF be the set of propositional formulas containing no other parameters but $0, 1$ and $\rightarrow$. Complete the equations below such that they yield procedures that compute the following functions:

$$\begin{aligned} &\mathit{impl} \in \mathrm{PT} \rightarrow \mathrm{PT} \rightarrow \mathrm{PT} \qquad && \mathit{pt} \in \mathrm{MF} \rightarrow \mathrm{PT} \\ &\mathit{impl}\ s\ t = \pi(s \rightarrow t) && \mathit{pt}\ s = \pi\ s \end{aligned}$$

$$\mathit{red}\ x\ s\ t \ = \ \text{if } s = t \text{ then } s \text{ else } (x, s, t)$$

$\mathit{impl}\ 0\ t \ =$

$\mathit{impl}\ 1\ t \ =$

$\mathit{impl}\ s\ 1 \ =$

$\mathit{impl}\ (x, s_0, s_1)\ 0 \ =$

$\mathit{impl}\ (x, s_0, s_1)\ (y, t_0, t_1) \ =$

if $x = y$ then

else if $x < y$ then

else

$\mathit{pt}\ 0 \ =$

$\mathit{pt}\ 1 \ =$

$\mathit{pt}\ x \ =$

$\mathit{pt}\ (s \rightarrow t) \ =$