Assignment 1
Introduction to Computational Logic, SS 2008

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Read in the lecture notes: Chapter 1

In all of the exercises you are free to use lambda notation whenever you think this is necessary.

Exercise 1.1 (Boolean Statements) Decide whether the following statements are valid for all values of the variables $x, y, z \in \mathbb{B}$. (Recall from class that $\mathbb{B} = \{0, 1\}$.) In case a statement is not valid, find values for the variables for which it does not hold.

a) $1 \to x \;\equiv\; x$

b) $(x \to y) \to (\neg y \to \neg x) \;\equiv\; 1$

c) $x \wedge y \vee \neg x \wedge z \;\equiv\; y \vee z$

Exercise 1.2 (Boolean Connectives) Consider the values

$$\begin{aligned} 0, 1 &\in \mathbb{B} \\ \neg &\in \mathbb{B} \to \mathbb{B} \\ \wedge, \vee, \to, \equiv &\in \mathbb{B} \to \mathbb{B} \to \mathbb{B} \end{aligned}$$

With 0 and $\to$ one can describe 1 as follows: $1 = (0 \to 0)$.

a) Describe $\neg$, $\wedge$, $\vee$, $\equiv$ with 0 and $\to$.

b) Describe 0, $\wedge$, and $\to$ with 1, $\neg$, and $\vee$.

Exercise 1.3 (Sets as Functions) Let X be a set. The subsets of X can be represented as the functions $X \to \mathbb{B}$. We use $P(X)$ as abbreviation for $X \to \mathbb{B}$. Express the following set operations with the logical operations $\neg$, $\wedge$, $\vee$, and $\forall_X$. To give you an idea what to do, here is how one would express set intersection $\cap \in P(X) \to P(X) \to P(X)$: $\cap = \lambda f{\in}P(X).\, \lambda g{\in}P(X).\, \lambda x{\in}X.\, fx \wedge gx$.

a) Union $\cup \in P(X) \to P(X) \to P(X)$

b) Difference $- \in P(X) \to P(X) \to P(X)$

c) Subset $\subseteq \in P(X) \to P(X) \to \mathbb{B}$

d) Disjointness $\parallel \in P(X) \to P(X) \to \mathbb{B}$

e) Membership $(\in) \in X \to P(X) \to \mathbb{B}$

Exercise 1.4 (Identities and Quantifiers) Express

a) $\forall_X$ with $=_{X\to\mathbb{B}}$ and 1.

b) $\exists_X$ with $\forall_X$ and $\neg$.

c) $\forall_X$ with $\exists_X$ and $\neg$.

d) $=_{X\to Y}$ with $\forall_X$ and $=_Y$.

e) $=_{\mathbb{B}}$ with $\equiv$.

f) $=_X$ with $\forall_{X\to\mathbb{B}}$ and $\to$.

Exercise 1.5 (Henkin's Reduction) In a paper published in 1963, Leon Henkin expressed the Boolean operations and the quantifiers with the identities.

a) Express 1 with $=_{\mathbb{B}\to\mathbb{B}}$.

b) Express 0 with 1 and $=_{\mathbb{B}\to\mathbb{B}}$.

c) Express $\neg$ with 0 and $=_{\mathbb{B}}$.

d) Express $\forall_X$ with 1 and $=_{X\to\mathbb{B}}$.

e) Express $\wedge$ with 1, $=_{\mathbb{B}}$, and $\forall_{\mathbb{B}\to\mathbb{B}\to\mathbb{B}}$.

f) Express $\wedge$ with 1 and $=_{(\mathbb{B}\to\mathbb{B}\to\mathbb{B})\to\mathbb{B}}$.

Exercise 1.6 (Choice Function) Let $\mathbf{C} \in (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$ be a function that satifies $f(\mathbf{C}f) = \exists f$ for all $f \in \mathbb{N} \to \mathbb{B}$. Furthermore, let the Boolean operations $\neg$, $\wedge$, $\vee$, $\to$, addition $+ \in \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ and the identity $=_{\mathbb{N}}$ be given. Since 0 is the only number $x \in \mathbb{N}$ such that $x + x = x$, it can be described with the given primitives as follows: $0 = \mathbf{C}(\lambda x{\in}\mathbb{N}.\ x + x = x)$. Describe the following values in an analogous way.

a) $f \in \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ such that $fxy = x - y$ if $x \geq y$

b) The existential quantifier $\exists \in (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$

c) The less or equal test $\leq\ \in \mathbb{N} \to \mathbb{N} \to \mathbb{B}$

d) $\mathit{max} \in \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ such that $\mathit{max}\ x\ y$ yields the maximum of x, y

e) $\mathit{if} \in \mathbb{B} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ such that $\mathit{if}\ b\ x\ y$ yields x if $b = 1$ and y otherwise

Exercise 1.7 (Tree Representations) Draw the locally nameless tree representations of the following terms.

a) $\lambda x{\in}\mathbb{B}.\ \lambda y{\in}\mathbb{B}.\ \neg x \vee y$

b) $\lambda x{\in}X.\ f(\lambda y{\in}Y.\ gyx)xy$

c) $(\forall x{\in}X\colon fx \wedge gx) \equiv \forall f \wedge \forall g$