Assignment 12
Introduction to Computational Logic, SS 2008

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Read in the lecture notes: Chapter 9

Exercise 12.1 (Herbrand Models) In his dissertation submitted in 1929, Jacques Herbrand introduces syntactic models for first-order logic that are now known as Herbrand models. Kripke sets are in fact Herbrand models for modal logic. Herbrand models are interesting for first-order logic since a first-order formula is satisfiable if and only if it is satisfied by a Herbrand model. Herbrand models can be constructed with the tableau method.

a) A Herbrand model for propositional logic is a set of Boolean variables. Let H be a set of Boolean variables. Define the evaluation function $\hat{H}$ that assigns to every propositional formula a Boolean value. Hint: Follow the definition of the evaluation function for Kripke sets and the grammar for propositional formulas.

b) Explain how the tableau method can be used to construct finite Herbrand models for satisfiable propositional formulas.

c) Find a Herbrand model for the propositional formula $\neg x \wedge (x \to y \vee z)$.

d) A Herbrand model for PLN is a set of PLN-formulas of the form $px_1 \dots x_n$ where $n \geq 0$. Let H be such a set. Define the evaluation function $\hat{H}$ that assigns to every PLN-formula a Boolean value.

e) Find finite Herbrand models for the following PLN-formulas:

   (i) $rxy \wedge qx \wedge (rxy \to rxx)$

   (ii) $\forall x \exists y.\, rxy$

Exercise 12.2 Prove the following interderivability facts:

- **Contra** $A \dot{\vdash} s \parallel A, \neg s \dot{\vdash} \bot$
- **Ded** $A \dot{\vdash} s \to t \parallel A, s \dot{\vdash} t$
- **Gen** $A \dot{\vdash} \forall x.s \parallel A \dot{\vdash} s$ if $x \notin \mathcal{N}A$
- **Rep** $A, s \dot{\vdash} \bot \parallel A, t \dot{\vdash} \bot$ if $A \vdash s{=}t$
- **Rep** $A \dot{\vdash} s \parallel A \dot{\vdash} t$ if $A \vdash s{=}t$

Exercise 12.3 Show the following facts about the closure $[\mathrm{PLN}]_\emptyset$.

a) $\bot$ is in $[\mathrm{PLN}]_\emptyset$.

b) For all $s \in PLN$, $\neg s \in [\mathrm{PLN}]_\emptyset$. (Hint: Do induction on s.)

c) $[\mathrm{PLN}]_\emptyset$ is closed under $\to$.

d) Prove that $\forall x.s$ is in the closure if s is in the closure and $x : I$ is an individual name (possibly a parameter).

Exercise 12.4 Reduce the following problems to modal refutability (analogous to Proposition 9.6.2). Prove the correctness of your reductions.

a) $\mathrm{M}, A \vdash \neg tx$

b) $\mathrm{M}, A \vdash \neg\forall t$

c) $\mathrm{M}, A \vdash \exists t$

d) $\mathrm{M}, A \vdash t_1 \neq t_2 \qquad$ where $t_1, t_2 : IB$

e) $\mathrm{M}, A \vdash \neg rxy$

f) $\mathrm{M}, A \vdash \forall x.\, t_1x \to t_2x$

Exercise 12.5 Show that the following proof step for $\Box$ is derivable.

$$\frac{A, ty \vdash \bot}{A, rxy, \Box rtx \vdash \bot}$$

(Assume $M \subseteq A$.)

Exercise 12.6 Extend the tableau rules to modal formulas that are not negation-normal. Give the corresponding proof steps.

Exercise 12.7 Refute the following sets with modal tableau proofs.

a) $\Diamond r(q \mathbin{\dot\vee} \dot\neg q)x,\ \ \Box rpx,\ \ \Box r(\dot\neg p)x$

b) $\Box r(q \mathbin{\dot\wedge} \dot\neg q)x,\ \ (\Diamond r(\dot\neg p) \mathbin{\dot\vee} \Diamond rp)x$

c) $(\dot\neg(\Diamond r(q \mathbin{\dot\vee} \dot\neg q) \mathbin{\dot\equiv} \Box rp \mathbin{\dot\to} \Diamond rp))x$

Exercise 12.8 Prove the validity of the following modal formulas by reduction to modal refutability (Proposition 9.6.2) and modal tableaux.

a) $\forall(\Box r(p \mathbin{\dot\wedge} q) \mathbin{\dot\to} \Box rp \mathbin{\dot\wedge} \Box rq)$

b) $\forall(\Diamond rp \mathbin{\dot\vee} \Diamond rq \mathbin{\dot\to} \Diamond r(p \mathbin{\dot\vee} q))$

c) $\forall(\Diamond r(p \mathbin{\dot\vee} q) \mathbin{\dot\to} \Diamond rp \mathbin{\dot\vee} \Diamond rq)$

Exercise 12.9 Use the tableau rules to construct an evident set that contains the modal formula $((\Diamond rp \mathbin{\dot\wedge} \Diamond rq) \mathbin{\dot\wedge} \Box r(\dot\neg p \mathbin{\dot\vee} \dot\neg q))x$. Explain why the evident set gives you a Kripke set that satisfies the formula.