# Assignment 5
# Introduction to Computational Logic, SS 2012

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Read in the lecture notes: Chapter 5

**Exercise 5.1** Draw the nameless tree representation of the following term. $\lambda xyz{:}U.\, x \to (\forall u{:}x.\, z \to y)$

**Exercise 5.2** Decide for each pair of term notations whether the two terms are identical.

a) $\forall x{:}U.x \to x$ and $\forall y{:}U.y \to y$
b) $\lambda xy{:}U.x \to y \to x$ and $\lambda yx{:}U.y \to x \to y$
c) $\lambda xyz{:}U.x \to (\forall u{:}x.z \to y)$ and $\lambda yxz{:}U.y \to (\forall u{:}x.z \to x)$
d) $\lambda x{:}U.x$ and $\forall x{:}U.x$
e) $(\lambda xy{:}U.y)x$ and $(\lambda x{:}U.\lambda z{:}U.z)x$

**Exercise 5.3** Determine the free variables of the following terms.

a) $\lambda x : y.\lambda z{:}x.\, zx$
b) $\lambda x : y.\forall z{:}x.\, x \to x'$

**Exercise 5.4** Write a substitution function *subst* : *nat* → *ter* → *ter* → *ter*. Follow the definition of the substitution function for untyped lambda calculus and the definition of the free variable test.

**Exercise 5.5** Determine the normal forms of the following terms.

a) $(\lambda x{:}U.\lambda g{:}U \to U \to U.\ (\lambda f{:}U \to U.\forall x : U.fx)(gx))\,U$
b) $\lambda x{:}U.(\lambda f{:}x \to x \to x.\lambda yz{:}x.f(fyz)(fzy))(\lambda yz{:}x.z)$

**Exercise 5.6** Explain why terms have at most one normal form.

**Exercise 5.7** Explain why convertibility of terms is an equivalence relation.

**Exercise 5.8** Give a term of the untyped lambda calculus that has a normal form but does not terminate.

**Exercise 5.9** Explain the following.

a) If $\Gamma \vdash s : t$, then $t$ terminates.
b) If $\Gamma, x : u, \Gamma' \vdash s : t$, then $u$ terminates.

**Exercise 5.10** Derive the following judgements.

a) $\emptyset \Rightarrow \mathsf{T}_3 : \mathsf{T}_5$

b) $\emptyset \Rightarrow \lambda x : \mathsf{T}_0 .\, x : \forall x : \mathsf{T}_0 . \mathsf{T}_0$

c) $\emptyset \Rightarrow \lambda x : \mathsf{T}_0 .\, x : \mathsf{T}_0 \to \mathsf{T}_0$

d) $x : \mathsf{T}_0 \Rightarrow (\lambda x : \mathsf{T}_0 .\, x) x : \mathsf{T}_0$

e) $\emptyset \Rightarrow \mathsf{T}_3 : (\lambda x : \mathsf{T}_6 . x) \mathsf{T}_5$

**Exercise 5.11** Make sure that you can reconstruct and explain the typing rules Lam, App, and Fun.

**Exercise 5.12** Find terms $s, s', t, t'$ such that $\emptyset \vdash s : t$, $\emptyset \vdash s' : t'$, $s \rhd s'$ and $\emptyset \nvdash s : t'$.

**Exercise 5.13** Answer the following questions and explain your answers.

a) Is there a nonterminating term that reduces to a terminating term?

b) Is there an ill-typed term that reduces to a well-typed term?

c) Is there a well-typed term that reduces to an ill-typed term?

d) Suppose $\Gamma \vdash s : t$ and $t \rhd u$. Does this imply $\Gamma \vdash s : u$?

e) Is $\emptyset \Rightarrow \mathsf{T}_1 : \mathsf{T}_0$ derivable?

f) Is $\forall x : \mathsf{T}_0 . \forall y : x .\, y$ well-typed?

**Exercise 5.14** Explain the following type error.

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Definition T : Type := forall X : Type, X -> X.
Check fun f : T => f T.
% Error : Universe inconsistency.
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**Exercise 5.15** Derive the following judgements in the type theory with *Prop*.

a) $\emptyset \Rightarrow \forall X : \mathit{Prop}. X \to X : \mathit{Prop}$

b) $\emptyset \Rightarrow \lambda X : \mathit{Prop}. \lambda x : X . x : \forall X : \mathit{Prop}. X \to X$