Semantics of Programming languages
Wednsday 16 Jan. 2002

$\alpha, \gamma \quad app(S) = \gamma \circ \alpha(S)$

$$\begin{array}{lrcl} \text{Safty:} & acc^{\#} & \sqsupseteq & \alpha(acc) \\ & \gamma(acc^{\#}) & \supseteq & acc \end{array}$$

$X = F(X)$
$\langle X_1, ..., X_n \rangle = F(\langle X_1, ..., X_n \rangle)$
$F = op_1 \circ ... \circ op_n$
$F \mapsto F^{\#}, op \mapsto op^{\#}$
$\langle L, \subseteq \rangle \mapsto \langle L^{\#}, \sqsubseteq \rangle$

not "=" in:
$\alpha(S) \sqcap \ \ \alpha(S') \sqsupseteq \alpha(S \cap S')$
$\alpha(S) +^{\#} \alpha(S') \sqsupseteq \alpha(S + S')$

$$\begin{array}{lcllcl} f^{\#}_{n/2}(\alpha(S)) & \sqsupseteq & \alpha(f_{n/2}(S)) & & & \\ & \sqsupseteq & \alpha(f_{n/2}(S)) & \text{if} & \alpha(S') & = a \\ & & & & & = \alpha(S) \end{array}$$

$$\begin{array}{lll} \text{Say} & S \subsetneq S' \Rightarrow & f(S) \subsetneq f(S') \\ & & \alpha(f(S)) \subsetneq \alpha(f(S')) = a' \end{array}$$

$\Rightarrow f^{\#}_{n/2}(\alpha(S)) \sqsupseteq \alpha(f(s_\nu)) \quad$ where $\quad s_\nu =$ the largest set of stores

$\{1\} \xrightarrow{\alpha} \text{odd} \xrightarrow{\gamma} \{1, 3, 5, ...\}$
$\{1, 3, 5, ...\} \xrightarrow{\alpha} \text{odd}$

$$\begin{array}{ll} \Rightarrow f^{\#}_{n/2}(\alpha(S)) & \sqsupseteq \alpha(f(\gamma(a))) \\ & \sqsupseteq \alpha(f(\gamma(\alpha(S)))) \sqsupseteq \alpha(f(S')) \quad \text{where} \quad S' \xrightarrow{\alpha} a \end{array}$$

Best approximation $f^{\#}$ of f:
$f^{\#}(\alpha(S)) = \alpha f \gamma \alpha(S)$
$f^{\#}(a) = \alpha f \gamma(a) \quad$ for all $\quad a \in L^{\#}$("Abs")

$f^{\#}_{n/2} : \{\perp, \top, o, e\} \to \{\perp, \top, o, e\}$
$f^{\#}_{n/2} = \alpha f_{n/2} \gamma$

$$
\begin{array}{lll}
\alpha : S \mapsto & \bot & S = \emptyset \\
 & o & S \subseteq \{1,3,5,...\} \\
 & e & S \subseteq \{2,4,6,...\} \\
 & \top & \text{otherwise}
\end{array}
$$

$$
\begin{array}{llll}
\gamma : & \bot & \mapsto & \emptyset \\
 & o & \mapsto & \{1,3,5,...\} \\
 & e & \mapsto & \{2,4,6,...\} \\
 & \top & \mapsto & I\!\!N
\end{array}
$$

a	$\bot$	$\top$	o	e
$f^{\#}_{n/2}(a)$	$\bot$	$\top$	o	e

$f^{\#}_{n/2}(S) = \{x/2 | x \in S\}$

Sign analysis:
$+^{\#} : \{>, 0, <, \top\}^2 \rightarrow \{>, 0, <, \top\}$

$$
\begin{tikzcd}
2^{store} \arrow[r, "f"] \arrow[d, "\alpha"] & 2^{store} \arrow[d, "\alpha"] \\
L^{\#} \arrow[r, "f^{\#}"] & L^{\#}
\end{tikzcd}
\qquad
\begin{tikzcd}
2^{store} \arrow[r, "f"] \arrow[d] & 2^{store} \\
L^{\#} \arrow[r, "f^{\#}"] & L^{\#} \arrow[u, "\gamma"]
\end{tikzcd}
$$

no "=" possible

$$
\begin{tikzcd}
2^{store} \arrow[r, "f"] \arrow[d, "\gamma"] & 2^{store} \arrow[d, "\alpha"] \\
L^{\#} \arrow[r, "f^{\#}"] & L^{\#}
\end{tikzcd}
$$

("=" for best possible approximation)

$F^{\#} = op_1^{\#} \circ ... \circ op_n^{\#}$

$a = F^{\#}(a) \quad (a = acc^{\#})$

least solution
$\overset{?}{\Longrightarrow}$ $a \sqsupseteq \alpha(acc)$ where $acc = F(acc)$least solution
or: $\mathrm{lfp}F^{\#} \sqsupseteq \alpha(\mathrm{lfp}F)$

In general, we don't have $F^{\#} = \gamma \circ F \circ \alpha$

$$\begin{tikzcd}
2^{store} \arrow[r, "f_1"] \arrow[d, "\alpha"] & 2^{store} \arrow[r, "f_2"] & 2^{store} \arrow[d, "\alpha"] & \sqcap\!| \\
L^{\#} \arrow[r, "f_2^{\#}"] & L^{\#} \arrow[r, "f_2^{\#}"] & \sqcap\!| \; L^{\#} \arrow[r, dashed, no head] & {} \arrow[u, "\gamma"']
\end{tikzcd}$$

$$\begin{tikzcd}
2^{store} \arrow[r, "f_1"] & 2^{store} \arrow[r, "f_2"] \arrow[d, "\alpha"] & 2^{store} \arrow[d, "\alpha"] \\
L^{\#} \arrow[u, "\gamma"] \arrow[r, "f_1^{\#}"] & L^{\#} \arrow[r, "f_2^{\#}"] & \sqcup\!| \; L^{\#}
\end{tikzcd}$$

$\sqsubseteq$ partial order (po)
$\sqcup X$ least upper bound of X
cpo: complete po
increasing chain
lattice $\sqsubseteq, X \sqcap Y, X \sqcup Y$
complete lattice $\sqcap X, \sqcup Y$
pointwise $\dot{\sqsubseteq}, \dot{\bot} = \lambda x \in S : \bot$

f: L $\overset{w}{\longrightarrow} L$ monotone
$f : L \overset{c}{\longrightarrow} L$ continuous

$X_0 \sqsubseteq X_1 \sqsubseteq ..., f(\sqcup_i X_i) = \sqcup_. f(X_i)$

f: L $\overset{a}{\longrightarrow} L$ additive
if for X an arbitrary set $\rightarrow \sqcup X \in L$, then $f(\sqcup X) = \sqcup f(X)$

$\sqcap X = \sqcup Y \quad$ where $Y = \{x | \forall x' \in X : x \sqsubseteq x'\}$

fixpoint of f is $x = f(p)$

$\mathrm{lfp} f = \mu x$ where $x = f(x)$

$\mathrm{fix}^{\sqsupseteq}(f) = \{x \in L | x \sqsupseteq f(x)\}$

1. Taski theorem: lfb exist and $\mathrm{lfb} f = \sqcap \mathrm{fix}^{\sqsupseteq}(f)$

2. if f continuous, then lfb can be defined constructively, namely $\mathrm{lfb} f = \sqcup_i f^i(\perp)$. proof by induction.