Semantics of Programming Languages
Assignment 11

Andreas Podelski, Patrick Maier, Jan Schwinghammer
http://www.ps.uni-sb.de/courses/sem-ws01/

INFORMATIK

Hand in until February, 6th, 2002

Let $\mathbf{D} = \langle D, \bot, \top, \bigvee, \bigwedge \rangle$ be a complete lattice.

We say that the join *completely distributes* over meets in $\mathbf{D}$ if for all $x \in D$ and $Y \subseteq D$, $x \wedge \bigvee Y = \bigvee\{x \wedge y \mid y \in Y\}$.

Let L be a set. We call a function $\delta : L \to D$ a *partition* of $\mathbf{D}$ if it satisfies $\top = \bigvee_{l \in L} \delta(l)$ (cover property) and $\forall k, l \in L : k \neq l \Rightarrow \delta(k) \wedge \delta(l) = \bot$ (disjointness property).

Exercise 11.1 (5) Let $\mathbf{D} = \langle D, \leq, \bigvee, \bigwedge \rangle$ be a complete lattice where the join completely distributes over meets. Let δ be a partition of $\mathbf{D}$ indexed by some set L. We define the ordered set $\mathbf{D}^\# = \langle D^\#, \sqsubseteq \rangle$ as componentwise ordered dependent product, i. e., $D^\# = \prod_{l \in L}\{x \wedge \delta(l) \mid x \in D\} = \big\{y : L \to D \mid \forall l \in L : y(l) \in \{x \wedge \delta(l) \mid x \in D\}\big\}$ and for all $y, y' \in D^\#$, $y \sqsubseteq y'$ iff $\forall l \in L : y(l) \leq y'(l)$.

Prove that $\mathbf{D} \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathbf{D}^\#$ is a Galois connection, where α and γ are defined such that for all $x \in D$ and $l \in L$, $\alpha(x)(l) = x \wedge \delta(l)$ and for all $y \in D^\#$, $\gamma(y) = \bigvee_{l \in L} y(l)$.

Exercise 11.1 abstractly shows how to obtain an abstract domain as a partition of a concrete domain. Now, we are going to apply this exercise to develop a Galois connection between two familiar concrete domains.

We introduce stores for typed variables. Let *Var* be a set of variables, let *Type* be a set of sets of values and let $\mathit{type} : \mathit{Var} \to \mathit{Type}$ be a type assignment. We define the typed store $\mathit{Store} = \prod_{x \in \mathit{Var}} \mathit{type}(x)$ as dependent product, i. e., $\mathit{Store} = \{s : \mathit{Var} \to \bigcup \mathit{Type} \mid \forall x \in \mathit{Var} : s(x) \in \mathit{type}(x)\}$. We assume that *Var* contains a distinguished variable *pc*, the program counter, with $\mathit{type}(\mathit{pc}) = \mathit{PP}$, the set of program points.

Exercise 11.2 (5) Prove that $\mathbf{D} = \langle 2^{\mathit{Store}}, \subseteq, \bigcup, \bigcap \rangle$ is a complete lattice where the join completely distributes over meets. Find the natural partition $\delta : \mathit{PP} \to 2^{\mathit{Store}}$ and prove that it is a partition.

Exercise 11.3 (5) Prove that there is an ordered set $\mathbf{D}^\# = \langle D^\#, \sqsubseteq \rangle$ such that $\mathbf{D} \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathbf{D}^\#$ is a Galois connection, where α and γ are defined in such a way that for all $S \subseteq \mathit{Store}$ and $p \in \mathit{PP}$, $\alpha(S)(p) = \{s \in S \mid s(\mathit{pc}) = p\}$ and for all $d \in D^\#$, $\gamma(d) = \bigcup_{p \in \mathit{PP}} d(p)$.
Hint: Apply exercise 11.1.

Exercise 11.4 (5) Let $\mathit{Var}' = \mathit{Var} \setminus \{pc\}$ and $\mathit{Store}' = \prod_{x \in \mathit{Var}'} \mathit{type}(x)$. Find an order $\sqsubseteq'$ on $(2^{\mathit{Store}'})^{PP}$ such that the ordered sets $\langle D^{\#}, \sqsubseteq \rangle$ and $\langle (2^{\mathit{Store}'})^{PP}, \sqsubseteq' \rangle$ are isomorphic. Explicitly specify the isomorphism φ and prove that it is one.

Exercise 11.5 (5) Using exercises 11.3 and 11.4, construct a Galois connection $\langle 2^{\mathit{Store}}, \subseteq \rangle \underset{\alpha'}{\overset{\gamma'}{\leftrightarrows}} \langle (2^{\mathit{Store}'})^{PP}, \sqsubseteq' \rangle$. Prove that α' and γ' are order isomorphisms and $\gamma' = \alpha'^{-1}$.