Semantics of Programming Languages: Solution of Assignment 3

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Exercise 3.1: (15) We assume that we have a function $H : \text{bool} \rightarrow \text{bool}$ such that

$$H\ M \rightarrow^* \text{true if } M \text{ has NF}$$
$$H\ M \rightarrow^* \text{false if } M \text{ has no NF}$$

for any expression $M : \text{bool}$. Using $div = \text{fix } \lambda x.\ x$ we derive

$$G \stackrel{def}{=} \lambda x.\ \texttt{if H x then } \text{div } \texttt{else } \text{true}$$

Exploiting the fact that fix $G = G$ (fix G) yields

$$\text{fix } G = \texttt{if (H } (\text{fix } \texttt{G})) \texttt{ then } \text{div } \texttt{else } \text{true}$$

We now show by case analysis that we have a contradiction.

1. Assume that fix G has a NF. Then we have H (fix G) $\rightarrow^*$ true and therefore, fix $G = div$.
2. Assume that fix G has no NF. Then we have H (fix G) $\rightarrow^*$ false and therefore, fix $G = true$.

Both cases lead to a contradiction. Therefore, there is no such function H.

Exercise 3.2: (20)

(a)

$$\begin{aligned}
\text{case } x \text{ Inleft Inright} &\stackrel{\eta}{=} \text{case } x\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } x) \\
&\stackrel{\beta}{=} \text{case } x\ (\lambda x.\ (\lambda x.\ x)\ (\text{Inleft } x))\ (\lambda x.\ (\lambda x.\ x)\ (\text{Inright } x)) \\
&\stackrel{case_3}{=} (\lambda x.\ x)\ x \\
&\stackrel{\beta}{=} x
\end{aligned}$$

(b)

$$\begin{aligned}
f\ (\text{case } (g \circ \text{Inleft})\ (g \circ \text{Inright})) &\stackrel{case_3}{=} f\ (g\ x) \\
&\stackrel{\beta}{=} (\lambda y.\ f\ (g\ y))\ x \\
&= (f \circ g)\ x \\
&\stackrel{case_3}{=} \text{case } x\ (f \circ g \circ \text{Inleft})\ (f \circ g \circ \text{Inright})
\end{aligned}$$

(c)

$$
\begin{aligned}
(\lambda x.\ \text{case } x f\ g) \circ \text{Inleft} &\overset{\beta}{=} \lambda x.\ \text{case (Inleft } x)\ f\ g \\
&\overset{case_1}{=} \lambda x.\ f\ x \\
&\overset{\eta}{=} f
\end{aligned}
$$

Similar for Inright.

$$
\begin{aligned}
f' &= \lambda x.\ \text{case } x\ f\ g \\
h\ (\text{case } x\ f\ g) &\overset{helper}{=} h\ (\text{case } x\ (f' \circ \text{Inleft})\ (f' \circ \text{Inright})) \\
&\overset{(b)}{=} \text{case } x\ (h \circ f' \circ \text{Inleft})\ (h \circ f' \circ \text{Inright}) \\
&\overset{helper}{=} \text{case } x\ (h \circ f)\ (h \circ g) \\
f\ (\texttt{if M then N else P}) &\overset{\beta}{=} f\ (\text{case } (\lambda y.\ N)\ (\lambda y.\ P)) \\
&\overset{helper}{=} \text{case } M\ (f \circ (\lambda y.\ N))\ (f \circ (\lambda y.\ P)) \\
&\overset{\beta}{=} \text{case } M\ (\lambda x.\ f\ N)\ (\lambda x.\ f\ P) \\
&\overset{\beta}{=} \texttt{if M then f N else f P}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\lambda z.\ \text{case } N\ P &\overset{(ass)}{=} \lambda z.\ \text{case } (\lambda x.\ M\ (\text{Inleft } x))\ (\lambda x.\ M\ (\text{Inright } x)) \\
&= \lambda z.\ \text{case } (M \circ \text{Inleft})\ (M \circ \text{Inright}) \\
&\overset{case_3}{=} \lambda z.\ M\ z \\
&\overset{\eta}{=} M
\end{aligned}
$$

Exercise 3.3: (10)

$$
\begin{aligned}
\text{true} &= \texttt{Eq}?\ (\text{fix not})\ (\text{fix not}) \\
&= \texttt{Eq}?\ (\text{fix not})\ (\text{not (fix not)}) \\
&= \text{false}
\end{aligned}
$$

Exercise 3.4: (10)

(a)

$$
\begin{aligned}
\text{succ } \lceil n \rceil &\to^* (\lambda x.\ \text{up (Inright } x))\ \lceil n \rceil \\
&\to^* \text{up (Inright } \lceil n \rceil) \\
&\to^* \lceil n\ +\ 1 \rceil
\end{aligned}
$$

(b)

$$\begin{aligned}
\text{zero? } \lceil 0 \rceil &\to^* (\lambda x.\ \text{case } (\text{dn } x)\ (\lambda y.\ \text{true})\ (\lambda z.\ \text{false}))\ (\text{up } (\text{Inleft } *)) \\
&\to^* \text{case } (\text{dn } (\text{up } (\text{Inleft } *)))\ (\lambda y.\ \text{true})\ (\lambda z.\ \text{false}) \\
&\to^* \text{case } (\text{Inleft } *)\ (\lambda y.\ \text{true})\ (\lambda z.\ \text{false}) \\
&\to^* (\lambda y.\ \text{true})\ * \\
&\to^* \text{true} \\
\text{zero? } \lceil n \rceil &\to^* (\lambda x.\ \text{case } (\text{dn } x)\ (\lambda y.\ \text{true})\ (\lambda z.\ \text{false}))\ (\text{up } (\text{Inright } \lceil n-1 \rceil)) \\
&\to^* \text{case } (\text{dn } (\text{up } (\text{Inright } \lceil n-1 \rceil)))\ (\lambda y.\ \text{true})\ (\lambda z.\ \text{false}) \\
&\to^* \text{case } (\text{Inright } \lceil n-1 \rceil)\ (\lambda y.\ \text{true})\ (\lambda z.\ \text{false}) \\
&\to^* (\lambda z.\ \text{false})\ \lceil n-1 \rceil \\
&\to^* \text{false}
\end{aligned}$$

(c)

$$\begin{aligned}
\text{pred } (\text{succ } x) &\to^* (\lambda x.\ \text{case } (\text{dn } x)\ (\lambda y.\ 0)\ (\lambda z.\ z))\ ((\lambda x.\ \text{up } (\text{Inright } x))\ x) \\
&\to^* \text{case } (\text{dn } (\text{up } (\text{Inright } x)))\ (\lambda y.\ 0)\ (\lambda z.\ z) \\
&\to^* \text{case } (\text{Inright } x)\ (\lambda y.\ 0)\ (\lambda z.\ z) \\
&\to^* (\lambda z.\ z)\ x \\
&\to^* x
\end{aligned}$$

Exercise 3.5: (20)

(a)

$$\begin{aligned}
\text{nil} &= \text{up } (\text{Inleft } *) \\
\text{cons} &= \lambda x.\ \lambda l.\ \text{up } (\text{Inright } \langle x, l \rangle)
\end{aligned}$$

(b) Definition

$$\begin{aligned}
\text{car} &= \lambda x.\ \text{case } (\text{dn } x)\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } (Proj_1\ x)) \\
\text{cdr} &= \lambda x.\ \text{case } (\text{dn } x)\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } (Proj_2\ x))
\end{aligned}$$

Verification

$$\begin{aligned}
\text{car nil} &= (\lambda x.\ \text{case } (\text{dn } x)\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } (Proj_1\ x)))\ (\text{up } (\text{Inleft } *)) \\
&= \text{case } (\text{Inleft } *)\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } (Proj_1\ x)) \\
&= \text{Inleft } * \\
\text{car } (\text{cons } x\ l) &= (\lambda x.\ \text{case } (\text{dn } x)\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } (Proj_1\ x)))\ ((\lambda x.\ \lambda l.\ \text{up } (\text{Inright } \langle x, l \rangle))\ x \\
&= \text{case } (\text{Inright } \langle x, l \rangle)\ (\lambda x.\ \text{Inleft } x)\ (\lambda x.\ \text{Inright } (Proj_1\ x)) \\
&= \text{Inright } (Proj_1\ \langle x, l \rangle) \\
&= \text{Inright } x
\end{aligned}$$

Similar for cdr.

(c)

$$\begin{aligned}
F &= \lambda f\textbf{.}\ (\text{cons } n\ f) \\
L_n &= \text{fix } F \\
\text{car } L_n &= \text{car } (\text{cons } n\ (\text{fix } F)) \\
&= \text{Inright } n \\
\text{cdr } L_n &= \text{cdr } (\text{cons } n\ (\text{fix } F)) \\
&= \text{Inright fix } F \\
&= \text{Inright } L_n
\end{aligned}$$

Exercise 3.6: (10)

(a) M Variable. $M \in \text{Terms}^s(\Sigma, \Gamma) \Rightarrow M : s \in \Gamma$. $\Gamma \subseteq \Gamma'$ yields $M : s \in \Gamma'$. Thus $M \in \text{Terms}^s(\Sigma, \Gamma')$.

(b) $M = fM_1 \ldots M_k$. $M \in \text{Terms}^s(\Sigma, \Gamma)$, with $\Sigma = \langle S, F \rangle$. Therefore, $f : s_1 \times \ldots \times s_k \to s \in F$. By induction hypothesis, $M_1, \ldots, M_k \in \text{Terms}^{s_i}(\Sigma, \Gamma')$. This yields $fM_1 \ldots M_k \in \text{Terms}^s(\Sigma, \Gamma')$.

Exercise 3.7: (15)

(a) Let $\eta(x) = 3$ and $\eta(s) = \langle 2, 1 \rangle$.

$$\begin{aligned}
[\![\text{ push } x \text{ (pop } s)\ ]\!]\eta &= \text{push}^A([\![\ x\ ]\!]\eta, \text{pop}^A([\![\ s\ ]\!]\eta)) \\
&= \text{push}^A(3, \text{pop}^A(\langle 2, 1 \rangle)) \\
&= \text{push}^A(3, \langle 1 \rangle) \\
&= \langle 3, 1 \rangle
\end{aligned}$$

(b) Let $\eta(s) = n :: s', somen \in \mathbb{N}, s \in \mathbb{N}^*$.

$$\begin{aligned}
[\![\text{ push (top } s) \text{ (pop } s)\ ]\!]\eta &= \text{push}^A(\text{top}^A([\![\ s\ ]\!]\eta), \text{pop}^A([\![\ s\ ]\!]\eta)) \\
&= \text{push}^A(\text{top}^A(n :: s'), \text{pop}^A(n :: s')) \\
&= \text{push}^A(n, s') \\
&= n :: s'
\end{aligned}$$

Let $\eta(s) = \epsilon$.

$$\begin{aligned}
[\![\text{ push (top } s) \text{ (pop } s)\ ]\!]\eta &= \text{push}^A(\text{top}^A(\epsilon), \text{pop}^A(\epsilon)) \\
&= \text{push}^A(0, \epsilon) \\
&= \langle 0 \rangle
\end{aligned}$$