Semantics, WS 2003: Solutions for assigment 1

Prof. Dr. Gert Smolka, Dipl.-Inform. Guido Tack

Exercise 1.1: Church numerals – Power The most compact term looks like this:

`power = λm.λn.n m`

Let us introduce some notation: $s^n z$ means applying s n times to z: $s(s(s(\ldots(sz)\ldots)))$. To disambiguate this from ordinary exponentiation, n to the power of m will be written like this: $[n^m]$.

We prove by induction that the term given above really computes the factorial:

- Case 1: $n = 0$
  $(\lambda sz.z)(\lambda sz.s^m z) \to \lambda z.z$ (equivalent to 1 through η-expansion)
- Case 2: $n = 1$
  $(\lambda sz.sz)(\lambda sz.s^m z) \to \lambda z.(\lambda sz.s^m z)z$ (equivalent to m through η-reduction)
- Case 3: $n > 1$
  $(\lambda sz.s^n z)(\lambda sz.s^m z)$
  $\to \lambda z'.(\lambda sz.s^m z)^n z'$
  $\to \lambda z'.(\lambda sz.s^m z)^{n-1}((\lambda sz.s^m z)z')$
  $\to \lambda z'.(\lambda sz.s^m z)^{n-1}(\lambda z.z'^m z)$
  with the induction hypothesis it follows that
  $\to \lambda z'.(\lambda sz.s^{[m^{n-1}]}z)(\lambda z.z'^m z)$
  $\to \lambda z'.\lambda z.(z'^m)^{[m^{n-1}]}z$ (η-reduction)
  $\to \lambda z'.\lambda z.z'^{[m^n]}z$

Exercise 1.2: Church numerals – Subtraction TAPL, 5.2.5

Exercise 1.3: Church numerals – Equality TAPL, 5.2.7

Exercise 1.4: Church numerals – Factorial Define `fac` as follows:

```
zz = pair c1 c1
ss = λ p.pair (times (fst p) (snd p)) (scc (snd p))
fac = λ n.fst (n ss zz)
```

This function works by building $n + 1$ pairs, and pair number i has the form $(i!, i + 1)$. Illustration:

$$
\begin{array}{ccc}
(c_{(n-1)!} \times c_n, \mathrm{scc}\ c_n) & = & (n!, n+1) \\
\vdots & & \\
(c6 \times c4, \mathrm{scc}\ c4) & = & (24, 5) \\
(c2 \times c3, \mathrm{scc}\ c3) & = & (6, 4) \\
(c1 \times c2, \mathrm{scc}\ c2) & = & (2, 3) \\
(c1 \times c1, \mathrm{scc}\ c1) & = & (1, 2) \\
(c1, c1) & = & (1, 1)
\end{array}
$$

Exercise 1.5: Church numerals – Lists TAPL, 5.2.8

Exercise 1.6: Church numerals – Sum of list TAPL, 5.2.11