Assignment 8
Semantics, WS 2007/08

Prof. Dr. Gert Smolka, Dr. Jan Schwinghammer
www.ps.uni-sb.de/courses/sem-ws07/

Read Chapter 4 of the lecture notes and Chapter 20 of Pierce's TAPL

Exercise 8.1 (Conditional) Let $\mathcal{I}$ be an interpretation. Find a term that denotes in $\mathcal{I}$ the following function: $\lambda b{\in}\mathbb{B}.\ \lambda u{\in}\mathcal{I}X.\ \lambda v{\in}\mathcal{I}X.\ \mathit{if}\ b\ \mathit{then}\ u\ \mathit{else}\ v$.

a) Assume that $\mathcal{I}$ interprets the name C as choice for $\mathbf{B} \to X \to X \to X$.

b) Assume that $\mathcal{I}$ interprets the name C as choice for X.

Exercise 8.2 (Numbers) Let $\mathcal{I}$ be an interpretation that interprets the sort N and the names $+, \cdot : N \to N \to N$ as the natural numbers with addition and multiplication. In addition assume that $\mathcal{I}$ interprets $\mathrm{C} : (N \to \mathbf{B}) \to N$ as choice. Find terms that denote in $\mathcal{I}$ the following:

a) The number 0.

b) The number 1.

c) Subtraction $- : N \to N \to N$.

d) Less or equal $\leq : N \to N \to \mathbf{B}$.

e) Integer division $\div : N \to N \to N$. You can choose $x \div 0$ as you like.

Exercise 8.3 (Termination) Find a formula that is satisfied by an interpretation if and only if it interprets the name $r : T \to T \to \mathbf{B}$ with a terminating relation. A relation *terminates* if there is no infinite sequence $a_0, a_1, a_2, \dots$ such that the pair (a_i, a_{i+1}) is in the relation for all $i \in \mathbb{N}$.

Exercise 8.4 (Pairs) Let the following sorts and names be given:
$\mathit{pair} : X \to Y \to Z,\ \ \mathit{fst} : Z \to X,\ \ \mathit{snd} : Z \to Y$. Find a formula that is satisfied by an interpretation if and only if it interprets (up to isomorphism) Z with the cartesion product of the denotations of X and Y and *pair*, *fst*, *snd* as pairing and projection.

Exercise 8.5 (Binary trees) Give a type in FPC whose values represent binary trees where the nodes are marked with natural numbers.

Exercise 8.6 (Finitely branching trees) Give a type in FPC that corresponds to the following type of Standard ML: *datatype tree = T of tree list*.

Exercise 8.7 (Nontermination) Give a closed and well-typed term in FPC whose reduction doesn't terminate.

Exercise 8.8 (Recursion operator) Here is yet another recursion operator for the untyped λ-calculus: $\lambda f.\,(\lambda d.\,f(\lambda x.ddx))(\lambda d.\,f(\lambda x.ddx))$.

a) Translate this term into FPC such that it takes for given types S, T the type $((S \to T) \to S \to T) \to S \to T$.

b) Translate this term into a polymorphic procedure in Standard ML that doesn't employ procedural recursion.

Exercise 8.9 (Inhabitation) Show that for every type T of FPC there is a closed term t such that $\emptyset \vdash t : T$.

Exercise 8.10 (Embedding the untyped lambda calculus) Give a translation of the untyped λ-calculus with observable natural numbers into FPC. Do this by extending the definitions of Λ and τ that are given in the lecture notes. Explain what happens to the stuck untyped terms on the typed side.