Assignment 12
Semantics, WS 2007/08

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www.ps.uni-sb.de/courses/sem-ws07/

Read Chapters 16, 21.1-21.3 and 21.11 of Pierce's TAPL.

Until February 08, 2008, you will be able to evaluate this course at
https://www.clix.uni-saarland.de/

Exercise 12.1 (Algorithmic (sub-) typing) Consider the simply typed lambda calculus with the typing relation defined by the algorithmic typing rules.

a) Give a derivation of $\{l : Top\} \to (Top \to \{\}) <: \{l : Top\} \to (Top \to \{\})$, using the algorithmic subtyping rules.

b) Find t, t' and T such that $\emptyset \mapsto t : T$ and $t \to t'$, but where $\emptyset \mapsto t' : T$ does not hold. Why is this situation not in conflict with the Preservation Property?

Exercise 12.2 (Minimal typing) What is the minimal type of the following terms:

a) $\lambda x{:}Top.\{\}$

b) $\lambda x{:}T \to T'.\{\}$

c) *if true then false else* $\{\}$

d) *if true then* $\{l_1 = t_1, l_2 = t_2\}$ *else* $\{l_2 = t_2, l_3 = t_3\}$, where each t_i is a closed term of type T_i

Exercise 12.3 (Subtyping isorecursive types) Consider the simply typed lambda calculus (with subtyping and typing defined by the declarative typing rules).

a) Use the Amber rule to show that $S <: T$ implies $Tree_S <: Tree_T$, where $Tree_T := \mu X.1 + \{val : T, left : X, right : X\}$

b) Show that the following variant of the Amber rule is *not* sound.

$$\frac{\Sigma, X{<:}X \vdash S <: T}{\Sigma \vdash \mu X.S <: \mu X.T}$$

Exercise 12.4 (Equirecursive types as trees) Draw the (initial part of the possibly infinite) trees corresponding to the following type expressions:

- $T_1 := Top \to Top$
- $T_2 := \mu X.\mu Y.X \to (Y \times X)$
- $T_3 := \mu Z.Z \to (Z \times Z)$

Which subtyping relations $T_i <: T_j$ hold between them, with $<:$ being coinductively generated from the inference rules? Can any of these also be inferred with respect to the isorecursive interpretation of types, using the Amber rule?

Exercise 12.5 (Generating function for the typing relation) Consider the simply typed lambda calculus with procedure and record types, and let $\mathcal{U} = (Var \to_{fin} Ter) \times Ter \times Ty$. Give the generating function $F : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ that corresponds to the inference rules defining the relation $\{(\Gamma, t, T) \in \mathcal{U} \mid \Gamma \vdash t : T\}$.

Exercise 12.6 (Least and greatest fixed points) Let *Ter* be the set of (untyped) lambda calculus terms, and consider the functions $F, G : \mathcal{P}(Ter) \to \mathcal{P}(Ter)$ defined by

$$
\begin{aligned}
F(M) &:= \{t \mid \exists t' \colon\ t \to t' \ \wedge\ t' \in M\} \\
G(M) &:= \{t \mid \forall t' \colon\ t \to t' \ \Rightarrow\ t' \in M\}
\end{aligned}
$$

a) Prove that F and G are monotonic.

b) Show that the singleton set $\{\lambda x.x\}$ is G-consistent, but not G-closed. Is it F-consistent? F-closed?

c) Show that for $\Omega = (\lambda x.xx)(\lambda x.xx)$, $\{\Omega\}$ is F-consistent. Similarly, show that for any $\delta_f = \lambda x.f(xx)$ with f of the form $\lambda y.t$, the set $\{f^n(\delta_f \delta_f) \mid n \in \mathbb{N}\}$ is F-consistent. Are they F-closed?

d) Give explicit descriptions of the sets μF, νF, μG and νG.

e) Use part (c) and (d) to conclude, by co-induction, that Ω and $\delta_f \delta_f$ diverge.

Exercise 12.7 (Kleene Fixed Point Theorem) Let $\mathcal{U}$ be some set. A monotonic function $F : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ is called **continuous** if for all increasing chains $M_1 \subseteq M_2 \subseteq \ldots$ in $\mathcal{P}(\mathcal{U})$, $F(\bigcup_i M_i) = \bigcup_i F(M_i)$ holds.

a) Prove that if F is continuous then the sequence $(F^i(\emptyset))_{i \in \mathbb{N}}$ forms an increasing chain $\emptyset \subseteq F(\emptyset) \subseteq F(F(\emptyset)) \subseteq \ldots \subseteq F^i(\emptyset) \subseteq \ldots$.

b) Prove that if F is continuous then it has a least fixed point μF, given by $\mu F = \bigcup_i F^i(\emptyset)$.

c) Show that if $\mathcal{U}$ is finite and $F : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ is monotonic, then F is continuous. As a consequence note that, by (b), this provides us with a method to construct μF.

Exercise 12.8 (Existence of least fixed points) Give examples of sets $\mathcal{U}$ and functions $F : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ such that

a) F has no fixed point;

b) F has a fixed point, but no least fixed point;

c) F is not monotonic, but has a least fixed point;

d) F is monotonic but its least fixed point μF (which exists by the Knaster-Tarski Theorem) is *not* given by $\bigcup_i F^i(\emptyset)$.