Assignment 11
Semantics, WS 2009/10

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Send your solutions to Exercise 11.2 in a file named `lastname.v` to `doczkal@ps.uni-sb.de`. Make sure that the entire file compiles without errors. You can find a template file on the course webpage.

Recommended reading: Chapter 6 of the lecture notes.

Exercise 11.1 (Natural deduction) Derive the following formulas with the rules of natural deduction:

a) $(\forall x.\ (p \to q \to x) \to x) \to p \wedge q$

b) $p \to \forall x.\ (p \to x) \to (q \to x) \to x$

c) $(\forall x.\ (p \to x) \to (q \to x) \to x) \to p \vee q$

d) $p\,x \to \forall z.\ (\forall y.\ p\,y \to z) \to z$

Exercise 11.2 (Prawitz reduction) We express Prawitz's reduction in Coq as follows:

```
Definition bot := forall Z :Prop, Z.
Definition con (X Y :Prop) := forall Z :Prop, (X->Y->Z) -> Z.
Definition dis (X Y :Prop) := forall Z :Prop, (X->Z) -> (Y->Z) -> Z.
Definition ex (X :Type) (p :X->Prop) :=
    forall Z :Prop, (forall x :X, p x -> Z) -> Z.
```

a) Prove the following propositions, where $p \leftrightarrow q$ abbreviates $p \to q \wedge q \to p$.

   (i) *bot* $\leftrightarrow$ *False*

   (ii) $\forall p\,q.\ con\ p\ q \leftrightarrow p \wedge q$

   (iii) $\forall p\,q.\ dis\ p\ q \leftrightarrow p \vee q$

   (iv) $\forall X\ (p : X \to Prop).\ ex\ X\ p \leftrightarrow \exists x.\ p\,x$

b) Prove the following claims which show that the introduction and elimination rules for the coded disjunction can be derived.

```
Proposition dis_intro_left: forall X Y :Prop, X -> dis X Y.
Proposition dis_intro_right: forall X Y :Prop, Y -> dis X Y.
Proposition dis_elim:
    forall X Y Z :Prop, dis X Y -> (X->Z) -> (Y->Z) -> Z.
```

(The cases of the other encoded constants are similar, but not required in this exercise.)

Exercise 11.3 (Proof terms) Recall the proof terms from Assignment 7. Find proof terms for the following propositions, and verify your solutions with *exact* in Coq.

a) $\forall x{:}\mathsf{o}.\, x \to x$

b) $\forall x{:}\mathsf{o}.\, \forall y{:}\mathsf{o}.\, (\forall z{:}\mathsf{o}.\, (x \to y \to z) \to z) \to y$

c) $\forall p{:}\mathsf{o}.\, \forall q{:}\mathsf{o}.\, p \to \forall x{:}\mathsf{o}.\, (p \to x) \to (q \to x) \to x$

Exercise 11.4 (Typing rules for CC) Derive the following sequents in the calculus of constructions.

a) $\Rightarrow\ \forall x{:}\mathsf{o}.\, (\forall y{:}x.\, x)\ :\ \mathsf{o}$

b) $\Rightarrow\ \forall x{:}\mathsf{o}.\, \forall y{:}\mathsf{o}.\, x \to y \to x\ :\ \mathsf{o}$