Assignment 13
Semantics, WS 2011-2012

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Read Chapters 7 and 8 of the Lecture Notes

Note: This assignment is relevant for the Endterm.

Exercise 13.1 Give an example of a closed term t in the simply-typed lambda calculus such that there is no type T such that $R_T t$. Give an example of such a t that also terminates (relative to the nondeterministic weak reduction in Chapter 7 of the lecture notes).

Exercise 13.2 Consider the simply typed lambda calculus with the typing relation, the nondeterministic weak reduction, and the logical relation R in Chapter 7 of the lecture notes. Which of the following statements are true?

a) If $R_T t$, then $\emptyset \vdash t : T$.
b) If $R_T t$, then t terminates.
c) If $\emptyset \vdash t : T$, then $R_T t$.
d) If $\emptyset \vdash t : T$, then t terminates.
e) If $\Gamma \vdash t : T$ and θ is a closed substitution with the same domain as Γ, then $\emptyset \vdash \theta t : T$.
f) If $\Gamma \vdash t : T$ and $R_\Gamma \theta$, then $R_T(\theta t)$.
g) If $t \Rightarrow t'$ and $R_T t$, then $R_T t'$.
h) If $t \Rightarrow t'$ and $R_T t'$, then $R_T t$.

Remark: Exercises 13.3 - 13.9 concern the Calculus of Constructions (Chapter 8 of the Lecture Notes).

Exercise 13.3 Suppose $\emptyset \vdash s : t$ and s is normal. Find out whether s can be a variable or an application.

Exercise 13.4 Suppose the typing $\Gamma \vdash s : t$ is derivable and t reduces to u. Explain why the typing $\Gamma \vdash s : u$ is derivable.

Exercise 13.5 You can experiment with the typing rules in Coq. Do the following examples by hand (taking Type to be U_0) and check your results with Coq.

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Check fun (s : Type)  (t : s -> Type) => forall x : s, t x.
Check fun (s  u : Type)  (t : s -> u) => fun x : s => t x.
Check fun (u : Type)  (v : u -> Type) (s : forall x : u, v x)  (t  :  u) => s t.
Check fun X : Type => X -> forall X : Type, X.
```

Exercise 13.6 Derive the typing

$$\emptyset \vdash \lambda x : U_0.\lambda x : x.x \; : \; \forall x : U_0.\forall y : x.x$$

Exercise 13.7 Derive the following typings.

a) $X : U_0 \vdash (\lambda Y : U_0.Y)X : U_0$

b) $X : U_0 \vdash (\lambda X : U_0.X)X : U_0$

Exercise 13.8 Determine normal types of the following terms and check your results with Coq.

a) $\forall x : U_0.x$

b) $\lambda x : U_0.\forall y : U_0.x \to y$

c) $\lambda f : U_0 \to U_1.\forall x : U_0.f\, x$

d) $\lambda xyz : U_0.\lambda f : x \to y.\lambda g : y \to z.\lambda w : x.g\,(f\, w)$

Exercise 13.9 Convince yourself that the following terms are ill-typed.

a) $\forall x : U_0.\forall y : x.y$

b) $\lambda f : U_1 \to U_0.\forall x : f\, U_0.x$

c) $\lambda xyz : U_0.\lambda f : x \to y.\lambda g : y \to z.$
$\forall p : (x \to z) \to U_0.\; p(\lambda w : x.g\,(f\, w)) \to p(\lambda w : x.w)$