Semantics, WS 2011-2012: Solution for Assignment 13

Prof. Dr. Gert Smolka, Dr. Chad Brown

Note: This assignment is relevant for the Endterm.

Exercise 13.1 Give an example of a closed term t in the simply-typed lambda calculus such that there is no type T such that $R_T t$. Give an example of such a t that also terminates (relative to the nondeterministic weak reduction in Chapter 7 of the lecture notes).

Solution to Exercise 13.1 Any normal ill-typed t provides an example. One such example is $\lambda x : X.xx$.

Exercise 13.2 Consider the simply typed lambda calculus with the typing relation, the nondeterministic weak reduction, and the logical relation R in Chapter 7 of the lecture notes. Which of the following statements are true?

a) If $R_T t$, then $\emptyset \vdash t : T$.
b) If $R_T t$, then t terminates.
c) If $\emptyset \vdash t : T$, then $R_T t$.
d) If $\emptyset \vdash t : T$, then t terminates.
e) If $\Gamma \vdash t : T$ and θ is a closed substitution with the same domain as Γ, then $\emptyset \vdash \theta t : T$.
f) If $\Gamma \vdash t : T$ and $R_\Gamma \theta$, then $R_T(\theta t)$.
g) If $t \Rightarrow t'$ and $R_T t$, then $R_T t'$.
h) If $t \Rightarrow t'$ and $R_T t'$, then $R_T t$.

Solution to Exercise 13.2

a) True (by the definition of R – Lemma 7.4.1)
b) True (by the definition of R – Lemma 7.4.2)
c) True (by the Basic Lemma with the empty substitution – Lemma 7.4.10)
d) True (by the Termination result – Lemma 7.4.11)
e) This is false. Knowing θ is a closed substitution with the same domain as Γ is not strong enough. Here is a counterexample. Let Γ be $\emptyset^x_X$ and θ be $\emptyset^x_{\lambda z:X.z}$. We know $\Gamma \vdash x : X$, but $\emptyset \nvdash (\lambda z : X.z) : X$.
f) True (by the Basic Lemma – Lemma 7.4.10)

g) True by Lemma 7.4.6.

h) This is false because t may be ill-typed or may not be closed. Here is a counterexample. Let t be $(\lambda y : X.(\lambda x : X.x))y$. Since t is not closed, we do not have $R_T t$. On ther other hand, we do have $t \Rightarrow (\lambda x : X.x)$ and $R_{X\to X}(\lambda x : X.x)$.

Remark: Exercises 13.3 - 13.9 concern the Calculus of Constructions (Chapter 8 of the Lecture Notes).

Exercise 13.3 Suppose $\emptyset \vdash s : t$ and s is normal. Find out whether s can be a variable or an application.

Solution to Exercise 13.3 s cannot be a variable since the context is empty. s cannot be an application since it is normal and the context is empty.

Exercise 13.4 Suppose the typing $\Gamma \vdash s : t$ is derivable and t reduces to u. Explain why the typing $\Gamma \vdash s : u$ is derivable.

Solution to Exercise 13.4 By propagation $\Gamma \vdash t : U$ for some universe U. By preservation $\Gamma \vdash u : U$. By the conversion rule $\Gamma \vdash s : u$.

Exercise 13.5 You can experiment with the typing rules in Coq. Do the following examples by hand (taking Type to be U_0) and check your results with Coq.

```
Check fun (s : Type)  (t : s -> Type) => forall x : s, t x.
Check fun (s  u : Type)  (t :  s  -> u) => fun x : s => t x.
Check fun (u : Type)  (v : u -> Type) (s : forall x : u, v x)  (t  :  u) => s t.
Check fun X : Type => X -> forall X : Type, X.
```

Solution to Exercise 13.5

$$\forall s : U_0.(s \to U_0) \to U_0$$

```
forall  s  :  Type, (s  -> Type)  -> Type
```

$$\forall s : U_0.\forall u : U_0.(s \to u) \to s \to u$$

```
forall  s u :  Type, (s  -> u)  -> s  -> u
```

$$\forall u : U_0.\forall v : (u \to U_0).(\forall x : u.vx) \to \forall x : u.vx$$

```
forall  (u  :  Type) (v :  u  -> Type), (forall x :  u,  v x)  -> forall t  :  u,  v t
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$$U_0 \to U_1$$

```
Type  -> Type
```

Exercise 13.6 Derive the typing

$$\emptyset \vdash \lambda x : U_0 . \lambda x : x . x \ : \ \forall x : U_0 . \forall y : x . x$$

Solution to Exercise 13.6

$$\text{Lam}\ \dfrac{\text{Lam}\ \dfrac{\text{Var}\ \dfrac{\text{CV}\ \dfrac{\text{Var}\ \dfrac{\text{CV}\ \dfrac{\text{CE}\ \dfrac{}{\emptyset \vdash U_0 \ : \ U_1}}{x : U_0 \vdash U_0 \ : \ U_1}}{x : U_0 \vdash x \ : \ U_0}}{x : U_0, y : x \vdash U_0 \ : \ U_1}}{x : U_0, y : x \vdash y \ : \ x}}{x : U_0 \vdash \lambda x : x . x \ : \ \forall y : x . x}}{\emptyset \vdash \lambda x : U_0 . \lambda x : x . x \ : \ \forall x : U_0 . \forall y : x . x}$$

Exercise 13.7 Derive the following typings.

a) $X : U_0 \vdash (\lambda Y : U_0 . Y) X : U_0$

b) $X : U_0 \vdash (\lambda X : U_0 . X) X : U_0$

Solution to Exercise 13.7

$$\text{Ap}\ \dfrac{\text{Lam}\ \dfrac{\text{Var}\ \dfrac{\text{CV}\ \dfrac{\text{CV}\ \dfrac{\text{CE}\ \dfrac{}{\emptyset \vdash U_0 \ : \ U_1}}{X : U_0 \vdash U_0 \ : \ U_1}}{X : U_0, Y : U_0 \vdash U_0 \ : \ U_1}}{X : U_0, Y : U_0 \vdash Y \ : \ U_0}}{X : U_0 \vdash \lambda Y : U_0 . Y \ : \ U_0 \to U_0} \qquad \text{CV}\ \dfrac{\text{CE}\ \dfrac{}{\emptyset \vdash U_0 \ : \ U_1}}{X : U_0 \vdash X : U_0}}{X : U_0 \vdash (\lambda Y : U_0 . Y) X : U_0}$$

$$\text{Ap}\ \dfrac{\text{Lam}\ \dfrac{\text{Var}\ \dfrac{\text{CV}\ \dfrac{\text{CV}\ \dfrac{\text{CE}\ \dfrac{}{\emptyset \vdash U_0 \ : \ U_1}}{X : U_0 \vdash U_0 \ : \ U_1}}{X : U_0, Y : U_0 \vdash U_0 \ : \ U_1}}{X : U_0, Y : U_0 \vdash Y \ : \ U_0}}{X : U_0 \vdash \lambda X : U_0 . X \ : \ U_0 \to U_0} \qquad \text{CV}\ \dfrac{\text{CE}\ \dfrac{}{\emptyset \vdash U_0 \ : \ U_1}}{X : U_0 \vdash X : U_0}}{X : U_0 \vdash (\lambda X : U_0 . X) X : U_0}$$

Exercise 13.8 Determine normal types of the following terms and check your results with Coq.

a) $\forall x : U_0 . x$

b) $\lambda x : U_0 . \forall y : U_0 . x \to y$

c) $\lambda f : U_0 \to U_1 . \forall x : U_0 . f\, x$

d) $\lambda xyz : U_0 . \lambda f : x \to y . \lambda g : y \to z . \lambda w : x . g\, (f\, w)$

Solution to Exercise 13.8

a) U_1

b) $U_0 \to U_1$

c) $(U_0 \to U_1) \to U_1$

d) $\forall xyz : U_0 . (x \to y) \to (y \to z) \to x \to z$

Exercise 13.9 Convince yourself that the following terms are ill-typed.

a) $\forall x : U_0 . \forall y : x . y$

b) $\lambda f : U_1 \to U_0 . \forall x : f\, U_0 . x$

c) $\lambda xyz : U_0 . \lambda f : x \to y . \lambda g : y \to z .$
$\forall p : (x \to z) \to U_0 .\ p (\lambda w : x . g\, (f\, w)) \to p (\lambda w : x . w)$

Solution to Exercise 13.9

a) The Fun rule will fail because the type of y will be x which is not a universe.

b) The Fun rule will fail because the type of x will be $f U_0$ which is not a universe.

c) Let Γ be $x : U_0, y : U_0, z : U_0, f : x \to y, g : y \to z, p : (x \to z) \to U_0$. We do not have $\Gamma \vdash p(\lambda w : x.w)$ since the Ap rule will fail. The Ap rule will fail because p (in the context Γ) expects an argument of type $x \to z$ but $\lambda w : x.w$ has type $x \to x$.