Abstract Reduction Systems

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We study abstract reduction systems and obtain results concerning confluence and normalisation.

1 Relations

Given a type X, we call predicates $X \to X \to \mathbf{P}$ **relations**. The letters R and S will range over relations. Inclusion and equivalence of relations are defined as follows:

$$R \subseteq S \;:=\; \forall xy.\, Rxy \to Sxy$$
$$R \equiv S \;:=\; R \subseteq S \wedge S \subseteq R$$

Reflexivity, symmetry, transitivity, and functionality of relations are defined as follows:

$$\text{reflexive } R \;:=\; \forall x.\, Rxx$$
$$\text{symmetric } R \;:=\; \forall xy.\, Rxy \to Ryx$$
$$\text{transitive } R \;:=\; \forall xyz.\, Rxy \to Ryz \to Rxz$$
$$\text{functional } R \;:=\; \forall xyz.\, Rxy \to Rxz \to y = z$$

2 Reflexive Transitive Closure

Let $R : X \to X \to \mathbf{P}$. We define the **reflexive transitive closure** R^* of R as an inductive predicate:

$$\frac{}{R^*xx} \qquad\qquad \frac{Rxx' \qquad R^*x'y}{R^*xy}$$

We refer to the induction lemma for R^* as **star induction**. Moreover, we refer to the constructor mapping R to R^* as **star**.

Fact 1

1. R^* is reflexive and transitive.
2. *Expansion* $R \subseteq R^*$.
3. *Monotonicity* If $R \subseteq S$, then $R^* \subseteq S^*$.
4. *Completeness* If $R \subseteq S$ and S is reflexive and transitive, then $R^* \subseteq S$.
5. *Idempotence* $R^{**} \cong R^*$.

Proof Transitivity, monotonicity, and completeness follow with star induction. Idempotence is a straightforward consequence of expansion, completeness, and transitivity. ■

Note that Fact 1 tells us that R^* is the least reflexive and transitive relation containing R.

Exercise 2 There are several equivalent definitions of the reflexive transitive closure of a relation. Consider the inductive predicate $R^\#$ defined by the following rules:

$$\frac{Rxy}{R^\# xy} \qquad \frac{}{R^\# xx} \qquad \frac{R^\# xy \quad R^\# yz}{R^\# xz}$$

Prove $R^\# \cong R^*$.

Exercise 3 Define composition $R \circ S$ and powers R^n of relations and prove $R^{n+1} \cong R \circ R^n \cong R^n \circ R$ and $R^* xy \leftrightarrow \exists n.\, R^n xy$.

3 Basic Confluence

Let $R : X \to X \to \mathbf{P}$. We define:

$$\begin{aligned}
\text{joinable } Rxy &:= \exists z.\, Rxz \wedge Ryz \\
\text{diamond } R &:= \forall xyz.\, Rxy \to Rxz \to \text{joinable } Ryz \\
\text{confluent } R &:= \text{diamond } R^* \\
\text{semi-confluent } R &:= \forall xyz.\, Rxy \to R^* xz \to \text{joinable } R^* yz
\end{aligned}$$

Fact 4 (Diamond) R is semi-confluent if R satisfies the diamond property.

Proof Let R satisfy the diamond property. Let Rxy_1 and $R^* xy_2$ We show by induction on $R^* xy_2$ that y_1 and y_2 are joinable. If $x = y_2$, the claim is trivial. Otherwise, we have Rxx' and $R^* x' y_2$. By the diamond property, we have $Ry_1 u$ and $Rx'u$ for some u. By the inductive hypothesis for $R^* x' y_2$, we have joinable $R^* u y_2$. The claim follows. ■

Note that Fact 4 says that star preserves the diamond property (i.e., R^* satisfies the diamond property if R satisfies the diamond property).

Fact 5 (Semi-Confluence) R is confluent iff R is semi-confluent.

Proof The direction from confluence to semi-confluence is obvious. For the other direction, let R be semi-confluent, R^*xy_1 and R^*xy_2. We show by induction on R^*xy_1 that y_1 and y_2 are joinable. If $x = y_1$, the claim is trivial. Otherwise, let Rxx' and $R^*x'y_1$. By semi-confluence of R, we have $R^*x'u$ and R^*y_2u for some u. By the inductive hypothesis for $R^*x'y_1$, we have joinable R^*y_1u. The claim follows. ∎

We now define the prediamond property:

$$\text{prediamond } R \;:=\; \forall xyz.\, Rxy \to Rxz \to y{=}z \lor Ryz \lor Rzy \lor \text{joinable } R\ y\ z$$

The prediamond property is weaker than the diamond property but still implies semi-confluence.

Fact 6 (Prediamond)

1. If R satisfies the diamond property, then R satisfies the prediamond property.
2. If R satisfies the prediamond property, then R is confluent.

Proof Claim 1 is trivial. For Claim 2 it suffices by Fact 5 to show that R is semi-confluent. This follows with a straightforward adaption of the proof of Fact 4. ∎

Example 7 It does not seem possible to further relax the prediamond property without loosing confluence. For instance, $R12$, $R21$, $R10$ and $R23$ is a relation that is not confluent.

4 Evaluation and Normal Forms

Let $\succ : X \to X \to \mathbf{P}$. We define **reducible** and **normal** points as follows:

$$\text{reducible } x \;:=\; \exists y.\, x \succ y$$
$$\text{normal } x \;:=\; \neg\text{reducible } x$$

Normal points may also be called *irreducible* or *terminal* points.

The **evaluation relation** for $\succ$ is defined as follows:

$$x \rhd y \;:=\; x \succ^* y \land \text{normal } y$$

If $x \rhd y$, we say that x **evaluates** to y or that y is a **normal form** of x. Moreover, we say that x is **weakly normalizing** if it has a normal for.

We now show that confluence ensures uniqueness of normal forms.

Fact 8 If $x \succ^* y$ and x is normal, then $x = y$.

Proof Case analysis on $x \succ^* y$. ■

Fact 9 If $\succ$ is confluent, then $\triangleright$ is functional.

Proof Follows with Fact 8. ■

5 Step-Indexed Evaluator

A **step-indexed evaluator** for a reduction relation $\succ$ is a function $E : \mathsf{N} \to X \to \mathcal{O}(X)$ such that:

1. $x \triangleright t \;\leftrightarrow\; \exists n.\, Enx = {}^\circ y$.
2. $Enx = {}^\circ y \;\to\; E(\mathsf{S}n)x = {}^\circ y$.

The lambda calculus is a system for which one can construct a step-indexed evaluator. However, one cannot construct an unindexed interpreter $X \to \mathcal{O}(X)$ for lambda calculus since it is undecidable whether a term is normalizing.

We will obtain step-indexed interpreters from so-called reduction functions. A **reduction function for $\succ$** is a function $\rho : X \to X$ such that $x \succ^* \rho x$ and $x \triangleright y \to \exists n.\, \rho^n x = y$ for all x and y.

Fact 10 Let ρ be a reduction function. Then $\rho^n x = x$ if x is normal.

Proof By induction on n using Fact 8. ■

From $\rho x = x$ is does not necessarily follow that x is normal. For instance, we may have $x \succ x$ with x the single element of X.

Let normality for $\succ$ be decidable. We define a function

$$E : (X \to X) \to \mathsf{N} \to X \to \mathcal{O}(X)$$

satisfying the equations

$$\begin{aligned} E\rho 0x &= \emptyset \\ E\rho(Sn)x &= \text{if normal } x \text{ then } {}^\circ x \text{ else } E\rho n(\rho x) \end{aligned}$$

Fact 11 Let ρ be a reduction function for $\succ$ and let normality be decidable. Then:

1. If $\rho^n x$ normal, then $E\rho(Sn)x = {}^\circ\rho^n x$.
2. If $E\rho nx = {}^\circ y$, then $x \triangleright y$.
3. $E\rho$ is a step-indexed evaluator for $\succ$.

Proof Claim 1 follows by induction on n using Fact 10. Claim 2 follows by induction on n. Claim 3 follows with claims 1 and 2. ■

Exercise 12 Let ρ be a reduction function. Show that $x \triangleright y$ if and only if y is normal and $\rho^n x = y$ for some n.

6 Strong Normalisation

Informally, termination may be defined as the absence of infinite paths. This characterization doesn't say much constructively. However, there is an elegant inductive definition of termination that provides a strong induction lemma and thus works constructively.

We define an inductive predicate $\mathsf{SN}_R\, x$:

$$\frac{\forall y.\, Rxy \to \mathsf{SN}_R\, y}{\mathsf{SN}_R\, x}$$

We say that x is **strongly normalizing** in R if $\mathsf{SN}_R\, x$. Moreover, we say that R is **terminating** if $\mathsf{SN}_R\, x$ for every x.

Fact 13 $\mathsf{SN}_R\, x$ if x is normal in R.

Fact 14 (Unfolding) $\mathsf{SN}_R\, x \;\leftrightarrow\; \forall y.\, Rxy \to \mathsf{SN}_R\, y$.

Fact 15 (SN induction)
$(\forall x.\, \mathsf{SN}_R\, x \to (\forall y.\, Rxy \to py) \to px) \to (\forall x.\, \mathsf{SN}_R\, x \;\to px)$.

As one can see from the proposition formulating SN induction, the use of SN induction is very natural since it simply adds the inductive hypothesis for all successors to the proof goal. Incidentally, the induction lemma Coq generates for SN replaces the premise $\mathsf{SN}_R\, x$ with the equivalent premise $\forall y.\, Rxy \to \mathsf{SN}_R\, y$.

Fact 16 Let $\mathsf{SN}_R\, x$ and R^*xy. Then $\mathsf{SN}_R\, y$.

Proof By induction on R^*xy. ∎

Fact 17 (Morphism) Let R be a relation on X and S be a relation on A. Let $f : X \to A$ be a function such that $S(fx)(fy)$ whenever Rxy. Then $\mathsf{SN}_R\, x$ if $\mathsf{SN}_S(fx)$.

Proof Let $p := \lambda a.\forall x.\, fx{=}a \to \mathsf{SN}_R\, x$. We prove $\mathsf{SN}_S\, a \to pa$ for all a by induction on $\mathsf{SN}_S\, a$. We assume $\mathsf{IH} : \;\forall b.\, Sab \to pb$ and prove pa. We assume $fx = a$ and prove $\mathsf{SN}_R\, x$. By unfolding, we assume Rxy and prove $\mathsf{SN}_R\, y$. We have $Sa(fy)$ by the assumption. By IH, we have $p(fy)$. The claim $\mathsf{SN}_R\, y$ follows. ∎

The morphism lemma is very useful in practice. For instance, if one wants to show that $\mathsf{SN}(st)$ implies $\mathsf{SN}\, s$ in a λ-calculus, one can simply apply the morphism lemma with the morphism $fu := ut$.

Constructively, one cannot show in general that a strongly normalizing point has a normal form. A relation is **classical** if reducibility is logically decidable (i.e., a point is either reducible or irreducible).

Fact 18 In a classical relation, strongly normalizing points are weakly normalizing.

Proof By SN induction. ∎

Fact 19 (Transitive closure) Let $R^+xy := \exists x'.\, Rxx' \wedge R^*x'y$.
Then $\mathsf{SN}_R\, x \leftrightarrow \mathsf{SN}_{R^+}\, x$.

Proof Both directions follow by SN induction. We show the direction from left to right, the other direction is routine.

Let $\mathsf{SN}_R\, x$. We show $\mathsf{SN}_{R^+}\, x$ by induction on $\mathsf{SN}_R\, x$. By unfolding of $\mathsf{SN}_{R^+}\, x$ we assume R^+xy and prove $\mathsf{SN}_{R^+}\, y$. Case analysis.

1. Rxy. Thus $\mathsf{SN}_{R^+}\, y$ by the inductive hypothesis.
2. Rxx' and $R^+x'y$ for some x'. The $\mathsf{SN}_{R^+}\, x'$ by the inductive hypothesis. Thus $\mathsf{SN}_{R^+}\, y$ by unfolding. ∎

Exercise 20 Give a relation on $\{0, 1\}$ such that 0 has a normal form but is not strongly normalizing.

Exercise 21 Let $\mathsf{SN}_R\, x$. Prove $\neg Rxx$.

Exercise 22 Let $R \subseteq S$ and $\mathsf{SN}_S x$. Prove $\mathsf{SN}_R\, x$.

Exercise 23 A common but limited technique for proving that a relation R is terminating is to give a function f such that $Rxy \to fx > fy$ for all x and y.

a) Prove that the relation $m > n$ on N is terminating.

b) Prove that a relation R is terminating if there is a function $f : X \to \mathsf{N}$ such that $Rxy \to fx > fy$ for all x and y. Use the morphism lemma.

c) Consider the following inductively defined relation R on $\mathcal{O}(\mathsf{N})$:

$$\frac{}{R\ {}^\circ Sn\ {}^\circ n} \qquad\qquad \frac{}{R\ \emptyset\ {}^\circ n}$$

   i) Draw R as a graph.

   ii) Prove that R is terminating.

   iii) Prove that there is no function $f : \mathcal{O}(\mathsf{N}) \to \mathsf{N}$ such that $Rxy \to fx > fy$.

7 Newman's Lemma

We define **local confluence** of relations as follows:

$$\text{locally confluent } R \ :=\ \forall xyz.\, Rxy \to Rxz \to \text{joinable } R^*\, y\, z$$

Clearly, relations satisfying the prediamond property are locally confluent. We may ask whether locally confluent relations are always confluent. It turns out that there are finite locally confluent relations that are not confluent. Example 7 provides such a counterexample. On the positive side, we can show that every terminating relation is confluent if it is locally confluent. This fact is known as Newman's lemma. There is an elegant proof of Newman's lemma using SN induction.

Fact 24 (Well-founded induction) Let R be terminating. Then px if $\forall x.\ (\forall y.\ Rxy \to py) \to px$.

Proof Follows with Fact 15. ∎

Fact 25 (Newman's Lemma)
Let R be terminating and locally confluent. Then R is confluent.

Proof Let $px := \forall yz.\ R^*xy \to R^*xz \to \exists u.\ R^*yu \wedge R^*zu$. It suffices to prove px for all x. By well-founded induction we have the inductive hypothesis $\forall y.\ Rxy \to py$.

By definition of p we assume R^*xy and R^*xz and prove that y and z are joinable in R^*. If $x = y$ or $x = z$, the claim is trivial. Otherwise we have Rxx_1, R^*x_1y, Rxx_2, and R^*x_2z. We will use the inductive hypothesis for both x_1 and x_2.

By local confluence we have R^*x_1u and R^*x_2u for some u. By the inductive hypothesis for x_2 we have some v such that R^*uv and R^*zv. By the inductive hypothesis for x_1 and transitivity of R^* we have R^*yw and R^*vw for some w. Joinability of y and z now follows by transitivity of R^*. ∎

8 Uniform Confluence

We define **uniform confluence** of relations as follows:

$$\text{uniformly\_confluent } R \ := \ \forall xyz.\ Rxy \to Rxz \to y{=}z \vee \text{joinable } R\ y\ z$$

Fact 26

1. Every functional relation is uniformly confluent.
2. Every relation satisfying the diamond property is uniformly confluent.
3. Every uniformly confluent relation satisfies the prediamond property and thus is confluent.

We will show that for a uniformly confluent relation all reductions of a given point to a normal form have the same length.

We define **graded reduction** R^nxy as follows:

$$\frac{}{R^0xx} \qquad\qquad \frac{Rxx' \qquad R^nx'y}{R^{Sn}xy}$$

Fact 27

1. $R^n \subseteq R^*$ and $R^1 \cong R$.
2. If R^*xy, then R^nxy for some n.
3. If R^mxy and R^nyz, then $R^{m+n}xz$.
4. If R^nxy and x is normal in R, then $n = 0$ and $x = y$.

We define **graded joinability** as follows:

$$\text{graded\_joinable } Ryzmn \;:=\; \exists ukl.\, R^kyu \wedge R^lzu \wedge m+k = n+l \wedge k \leq n \wedge l \leq m$$

Fact 28 The following statements are equivalent:

1. R uniformly confluent.
2. $\forall xyzn.\, Rxy \to R^nxz \to \text{graded\_joinable } Ryz1n$.
3. $\forall xyzmn.\, R^mxy \to R^nxz \to \text{graded\_joinable } Ryzmn$.

Proof We prove $1 \to 2 \to 3 \to 1$. The implication $3 \to 1$ is straightforward.

- $1 \to 2$. The proof refines the proof of Fact 4. Assume (1) and Rxy and R^nxz. We prove by induction on n that R^kyu and R^lzu for some u, $k \leq n$ and $l \leq 1$ such that $1 + k = n + l$. If $n = 0$, then $x = z$ and the claim follows with $u = y$ and $l = 1$. Otherwise, we have Rxx' and $R^{n-1}x'z$. Case analysis using (1).
  - $y = x'$. The claim follows with $u = z$, $k = n - 1$, and $l = 0$.
  - Rxv and $Rx'v$ for some v. By the inductive hypothesis, we obtain u, $k' \leq n - 1$ and $l' \leq 1$ such that $R^{k'}vu$, R^lzu, and $1 + k' = n - 1 + l$. The claim follows with $k = 1 + k'$.
- $2 \to 3$. The proof refines the proof of Fact 5, direction semi-confluence to confluence. Assume (2) and R^mxy and R^nxz. We prove byinduction on m that R^kyu and R^lzu for some u, $k \leq n$ and $l \leq 1$ such that $m + k = n + l$. If $m = 0$, then $x = y$ and the claim follows with $u = z$, $k = n$, and $l = 0$. Otherwise, we have Rxx' and $R^{m-1}x'y$. By (2) we obtain v, $l' \leq 1$, and $k' \leq n$ such that $R^{k'}x'v$, $R^{l'}zv$, and $1 + k' = n + l'$. By the inductive hypothesis for $R^{m-1}x'y$ we obtain u, $k \leq k'$, and $l'' \leq m - 1$ such that R^kyu, $R^{l''}vu$, and $m - 1 + k = k' + l''$. The claim follows with $l = l' + l''$. ∎

Fact 29 (Uniform normalization) Let R be uniformly confluent, R^mxy, R^nxz, and z be normal for R. Then $m \leq n$ and $R^{n-m}yz$.

Proof Follows with Facts 28 and 27. ∎

Fact 30 Let R be uniformly confluent. Then every point that has a normal form is strongly normalizing.

Proof Let R^*xy and y be normal. Then $R^n xy$ for some n by Fact 27. We prove $\mathsf{SN}_R\, x$ by induction on n. For $n = 0$, we have $x = y$ and the claim follows by Fact 13. Otherwise, we have Rxx' and $R^{n-1}x'y$ for some x'. By unfolding of the claim, we assume Rxx'' and prove $\mathsf{SN}_R\, x''$. By the inductive hypothesis, it suffices to show $R^{n-1}x''y$, which follows by uniform normalization (Fact 29). ∎

Fact 31 Let $\succ : X \to X \to \mathbf{P}$ be uniformly confluent and $\rho : X \to X$ satisfy $x \succ^* \rho x$ and $\rho x = x \to \mathsf{normal}\ x$ for x. Then ρ is a reduction function for $\succ$.

Exercise 32 Give a confluent relation $\succ$ and a function ρ satisfying the conditions of Fact 31 that is not a reduction function for $\succ$.

9 TMT Method

The confluence of λ-calculi where reduction is possible within abstractions can be shown with a clever method building on work of Tait, Martin-Löf, and Takahashi. We speak of the *TMT method*. The TMT method factorises in three part:

1. An abstract part not making assumptions about terms. The abstract part yields confluence and a reduction function.
2. An intermediate part using terms but keeping substitution abstract.
3. A concrete part dealing with the concrete substitution used.

We present the abstract part in the following. The key idea is to show the confluence of a relation R by identifying a suitable auxiliary relation S such that $R \subseteq S \subseteq R^*$ and S has the diamond property.

Fact 33 (Sandwich) Let $R \subseteq S \subseteq R^*$. Then:

1. $R^* \cong S^*$.
2. If S has the diamond property, then R is confluent.

Proof Claim 1 follows with monotonicity and idempotence of star. Claim 2 follows with (1) and the fact that star preserves the diamond property. ∎

A **Takahashi function for $\succ$** is a function $\rho : X \to X$ such that $x \succ y \to y \succ \rho x$ for all x and y.

Fact 34 (Takahashi) Let ρ be a Takahashi function for $\succ$. Then:

1. *Diamond* If $x \succ y_1$ and $x \succ y_2$, then $y_1 \succ \rho x$ and $y_2 \succ \rho x$.
2. *Soundness* If $\succ$ is reflexive, then $x \succ \rho x$.
3. *Preservation* If $x \succ y$, then $\rho x \succ \rho y$.
4. *Cofinality* If $x \succ^* y$, then $y \succ^* \rho^n x$ for some n.

Proof Claim 1: Immediate from the Takahashi property of ρ.

Claim 2: Given x, we have $x \succ x$ by reflexivity. Thus $x \succ \rho x$.

Claim 3: Let $x \succ y$. Then $y \succ \rho x \succ \rho y$.

Claim 4: Let $x \succ^* y$. We prove $\exists n.\ y \succ^* \rho^n x$ by induction on $x \succ^* y$. The first subgoal is trivial. In the second subgoal we have $x \succ x' \succ^* y \succ^* \rho^n x'$ for some n using the inductive hypothesis. By Claim 3 we have $\rho^n x \succ \rho^n x'$. Thus $\rho^n x' \succ \rho(\rho^n x)$. Hence $y \succ^* \rho^{Sn} x$. ∎

Theorem 35 (TMT) Let $\succ$ and $\gg$ be predicates $X \to X \to \mathbf{P}$ such that $\succ \subseteq \gg \subseteq \succ^*$ and $\gg$ is reflexive. Moreover, let ρ be a Takahashi function for $\gg$. Then $\succ$ is confluent and $E\rho$ is a step-indexed evaluator for $\succ$.

Proof Follows with Facts 33, 34, 8, and 11. ∎

10 Equivalence Closure

General λ-calculi are best understood as deductive systems where one or several reduction rules generate an equivalence relation $s \equiv t$ on terms. If the reduction relation $s \succ t$ generated by the rules is confluent, as is typically the case for λ-calculi, there is a beautiful and useful connection between equivalence (deduction) and reduction (computation) known as Church-Rosser property:

$$s \equiv t \ \to\ \exists u.\ s \succ^* u \wedge t \succ^* u$$

From the Church-Rosser property one obtains the rule

$$\mathsf{normal}\ t \to s \equiv t \to s \rhd t$$

which makes it possible to verify a computational claim $s \rhd t$ by showing the equivalence $s \equiv t$ by means of undirected equational reasoning. This is exploited for the verification of programming techniques for λ-calculus (e.g., encoding of inductive data types and recursion).

We first study the connection between reduction and equivalence in the abstract and then apply the results to abstract $\lambda\beta$.

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

We define the **symmetric closure** $R^{\leftrightarrow}$ of a relation R as follows:

$$R^{\leftrightarrow} \ := \ \lambda xy.\ Rxy \vee Ryx$$

We write $R^{\leftrightarrow *}$ for $(R^{\leftrightarrow})^*$ and call $R^{\leftrightarrow *}$ the **equivalence closure of R**.

Fact 36

1. $R \subseteq R^{\leftrightarrow}$ and $R^{\leftrightarrow}$ is symmetric.
2. $R^* \subseteq R^{\leftrightarrow *}$.
3. If R is symmetric, then R^* is symmetric.
4. $R^{\leftrightarrow *}$ is an equivalence relation.
5. If $R \subseteq S$ and S is an equivalence relation, then $R^{\leftrightarrow *} \subseteq S$.

Note that Fact 36 tells us that $R^{\leftrightarrow *}$ is the least equivalence relation containing R. We define the **Church-Rosser property** for relations as follows:

$$\text{Church-Rosser } R \ := \ \forall xy.\, R^{\leftrightarrow *}xy \to \text{joinable } R^* xy$$

Fact 37 R is Church-Rosser if and only if R is confluent.

Proof The direction from Church-Rosser to confluence is obvious since $R^* \subseteq R^{\leftrightarrow *}$ (monotonicity of star).

For the other direction we assume that R is semi-confluent and that $R^{\leftrightarrow *}xy$. We show joinable R^*xy by star induction on $R^{\leftrightarrow *}xy$. If $x = y$, the claim is trivial. Otherwise, let $R^{\leftrightarrow}xx'$ and $R^{\leftrightarrow *}x'y$. We have $R^*x'z$ and R^*yz for some z by the inductive hypothesis. Case analysis on $R^{\leftrightarrow}xx'$. If Rxx', the claim follows. Otherwise, we have $Rx'x$. By semi-confluence of R we obtain some u such that R^*xu and R^*zu. Thus R^*yu by transitivity. The claim follows. ∎

Fact 38 Let R be a confluent relation, and let $x \rhd y$ and $x \equiv y$ be the accompanying evaluation and equivalence relations. Then:

1. $x \equiv y \to \text{normal } y \to x \rhd y$.
2. $x \equiv y \to \text{normal } x \to \text{normal } y \to x = y$.
3. $x \equiv y \to (x \rhd z \leftrightarrow y \rhd z)$.
4. $x \rhd a \to y \rhd b \to a \neq b \to x \not\equiv y$.

Proof Follows with Facts 37 and 8. ∎

Exercise 39 There are several equivalent definitions of the equivalence closure of a relation. Consider the inductive predicate $\equiv_R$ defined by the following rules:

$$\frac{Rxy}{x \equiv_R y} \qquad \frac{}{x \equiv_R x} \qquad \frac{x \equiv_R y}{y \equiv_R x} \qquad \frac{x \equiv_R y \qquad y \equiv_R z}{x \equiv_R z}$$

Prove $\equiv_R \cong R^{\leftrightarrow *}$.

11 Historical Remarks

The study of abstract reduction systems originated with Newman [6]. The notions of confluence, semi-confluence, diamond property, and uniform confluence all appear in Newman [6] (see Theorems 1–3). The modern, relational view of abstract reduction systems is due to Huet [4]. Baader and Nipkow's textbook [1] on term rewriting starts with a chapter on abstract reduction systems.

Newman's lemma appears as Theorem 3 in [6]. The constructive proof based on well-founded induction is due to Huet [4]. Newman's original proof is not constructive and does not employ well-founded induction. Newman's lemma was one of the first proofs done with Coq [2].

Newman's lemma is an example of a result where the constructive proof is shorter and clearer than the classical proof. Newman defined termination as the absence of infinite paths. That well-founded induction is valid for relations disallowing infinite paths was first observed by Emmy Noether. This fact can only be shown with excluded middle.

The inductive definition of strong normalization seems to originate with Coquand and Huet's [2] definition of noetherian relations and Huet's [4] use of well-founded induction (noetherian induction) for the proof of Newman's lemma. Newman [6], Huet [4], and Baader and Nipkow [1] define strong normalization as the absence of infinite paths, thus foregoing a constructive proof of well-founded induction.

The formulation of the abstract TMT method presented here is based on the literature on λ-calculus. There the sandwiched relation S is known as parallel reduction (see Hindley and Seldin [3], Appendix A2). Takahashi functions originated with Takahashi's [9] confluence proof for the lambda calculus.

The name uniform confluence is from [8, 7]. Niehren [7] observes that the call-by-value λ-calculus is uniformly confluent. Dal Lago and Martini [5] prove Fact 28 for the call-by-value λ-calculus.

References

[1] Franz Baader and Tobias Nipkow. *Term rewriting and all that.* Cambridge University Press, 1998.

[2] Thierry Coquand and Gérard Huet. Constructions: A higher order proof system for mechanizing mathematics. In *Proc. of EUROCAL'85, Linz, Austria*, volume 203 of *LNCS*, pages 151–184. Springer, 1985.

[3] J. Roger Hindley and Jonathan P. Seldin. *Lambda-Calculus and Combinators, an Introduction.* Cambridge University Press, 2008.

[4] Gérard Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *Journal of the ACM (JACM)*, 27(4):797–821, 1980.

[5] Ugo Dal Lago and Simone Martini. The weak lambda calculus as a reasonable machine. *Theor. Comput. Sci.*, 398(1-3):32–50, 2008.

[6] Maxwell Herman Alexander Newman. On theories with a combinatorial definition of "equivalence". *Annals of Mathematics*, 43(2):223–243, 1942.

[7] Joachim Niehren. Uniform confluence in concurrent computation. *J. Funct. Program.*, 10(5):453–499, 2000.

[8] Joachim Niehren and Gert Smolka. A confluent relational calculus for higher-order programming with constraints. In Jean-Pierre Jouannaud, editor, *CCL*, volume 845 of *LNCS*, pages 89–104. Springer, 1994.

[9] Masako Takahashi. Parallel reductions in λ-calculus. *Inf. Comput.*, 118(1):120–127, 1995.