

Reducing Theorem Proving to a Sequence of SAT Problems

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Introduction

- Directed, Cut-Free, Ground Tableau System for HOL (Brown, Smolka [LMCS 2010])
- Extended to include Choice (Backes, Brown [2010])
- Restricted Instantiations

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- Extended to include Choice (Backes, Brown [2010])
- Restricted Instantiations
- Prototype Implementation using MINISAT to Simulate the Tableau Calculus (2009-2010)

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(Best New System)

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- ...but how does Satallax work?
- This Talk: How to simulate a [first-order tableau calculus](#) with a sequence of SAT Problems.

First-Order Formulas

$$s, t ::= rxy \mid \neg rxy \mid s \vee t \mid s \wedge t \mid \forall x.s \mid \exists x.s$$

\bar{s} - negation normal form of the negation of s

Model \mathcal{M} (directed graph)

Assignment φ maps variables to vertices.

Satisfaction Relation: $\mathcal{M} \models_{\varphi} s$

First-Order Tableau Calculus

$$\mathcal{T}_{\vee} \frac{s \vee t}{s \mid t}$$

$$\mathcal{T}_{\wedge} \frac{s \wedge t}{s, t}$$

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$$\mathcal{T}_{\forall} \frac{\forall x.s}{s_y^x} \quad y \in \mathcal{U}$$

\mathcal{U} - restricted set of instantiations

First-Order Tableau Calculus

$$\mathcal{T}_{\vee} \frac{s \vee t}{s \mid t}$$

$$\mathcal{T}_{\wedge} \frac{s \wedge t}{s, t}$$

$$\mathcal{T}_{\forall} \frac{\forall x.s}{s_y^x} \quad y \in \mathcal{U}$$

$$\mathcal{T}_{\exists} \frac{\exists x.s}{s_y^x} \quad y \text{ FRESH}$$

\mathcal{U} - restricted set of instantiations

Example 1: First-Order Tableau Refutation

$$\forall xy. \neg rxy \vee ryy$$

$$\exists xy. \neg ryy \wedge rxy$$

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Branches

A **branch** is a finite sets of formulas (conjunctive)

$$B = \{s_1, \dots, s_n\}$$

$\mathcal{M} \models_{\varphi} B$ means $\mathcal{M} \models_{\varphi} s$ for every $s \in B$.

B is **satisfiable** if there is some \mathcal{M} and φ such that $\mathcal{M} \models_{\varphi} B$.

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- A branch B is **closed** if there is some $s \in B$ such that $\bar{s} \in B$.

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- A branch B is **closed** if there is some $s \in B$ such that $\bar{s} \in B$.
- A branch B is **open** if it is not closed.

Clauses

A **clause** is a finite set of formulas (disjunctive)

$$C = s_1 \sqcup \cdots \sqcup s_n$$

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- A set of clauses \mathcal{C} is **satisfiable** if there is some \mathcal{M} and φ such that $\mathcal{M} \models_{\varphi} C$ for every $C \in \mathcal{C}$.
- A set of clauses \mathcal{C} is **weakly satisfiable** if there is an open branch B such that $B \cap C \neq \emptyset$ for every $C \in \mathcal{C}$.

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- A set of clauses \mathcal{C} is **weakly satisfiable** if there is an open branch B such that $B \cap C \neq \emptyset$ for every $C \in \mathcal{C}$.
- Otherwise, \mathcal{C} is **strongly unsatisfiable**.

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A **clause** is a finite set of formulas (disjunctive)

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- A set of clauses \mathcal{C} is **satisfiable** if there is some \mathcal{M} and φ such that $\mathcal{M} \models_{\varphi} C$ for every $C \in \mathcal{C}$.
- A set of clauses \mathcal{C} is **weakly satisfiable** if there is an open branch B such that $B \cap C \neq \emptyset$ for every $C \in \mathcal{C}$.
- Otherwise, \mathcal{C} is **strongly unsatisfiable**.
- Weak satisfiability is equivalent to SAT.

Simulation of Example 1

Passive Formulas	Active Formulas	Passive Vars	Active Vars	Clauses
	<div data-bbox="484 246 518 280">1</div> $\forall xy. \neg rxy \vee ryy$ <div data-bbox="484 291 518 325">2</div> $\exists xy. \neg ryy \wedge rxy$			<div data-bbox="1115 215 1149 249">1</div> <div data-bbox="1115 260 1149 294">2</div>

Weakly Satisfiable:

1

,

2

Simulation of Example 1

Passive Formulas	Active Formulas	Passive Vars	Active Vars	Clauses
	<div data-bbox="484 246 518 282" style="border: 1px solid blue; display: inline-block; padding: 2px;">1</div> $\forall xy. \neg rxy \vee ryy$			<div data-bbox="1115 215 1149 251" style="border: 1px solid blue; display: inline-block; padding: 2px;">1</div>
	<div data-bbox="484 293 518 329" style="border: 1px solid blue; display: inline-block; padding: 2px;">2</div> $\exists xy. \neg ryy \wedge rxy$			<div data-bbox="1115 262 1149 298" style="border: 1px solid blue; display: inline-block; padding: 2px;">2</div>

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$\boxed{1} \forall xy. \neg rxy \vee ryy$	$\boxed{2} \exists xy. \neg ryy \wedge rxy$			$\boxed{1}$ $\boxed{2}$

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1 $\forall xy. \neg rxy \vee ryy$	2 $\exists xy. \neg ryy \wedge rxy$			1 2

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Weakly Satisfiable: $\boxed{1}, \boxed{2}, \boxed{3}$

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<div style="border: 1px solid black; padding: 2px; display: inline-block;">3</div> $\exists y. \neg ryy \wedge rxy$	<div style="border: 1px solid black; padding: 2px; display: inline-block;">6</div> rxy			<div style="border: 1px solid black; padding: 2px; display: inline-block;">2</div>
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Weakly Satisfiable:

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3

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6

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Weakly Satisfiable: 1, 2, 3, 4, -5, 6, 7

Simulation of Example 1

Passive Formulas	Active Formulas	Passive Vars	Active Vars	Clauses
<div style="border: 1px solid blue; padding: 2px; display: inline-block;">1</div> $\forall xy. \neg rxy \vee ryy$	<div style="border: 1px solid blue; padding: 2px; display: inline-block;">7</div> $\forall y. \neg rxy \vee ryy$	x	y	<div style="border: 1px solid blue; padding: 2px; display: inline-block;">1</div> <div style="border: 1px solid blue; padding: 2px; display: inline-block;">2</div> <div style="border: 1px solid blue; padding: 2px; display: inline-block;">-2</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">3</div> <div style="border: 1px solid blue; padding: 2px; display: inline-block;">-3</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">4</div> <div style="border: 1px solid blue; padding: 2px; display: inline-block;">-4</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">-5</div> <div style="border: 1px solid blue; padding: 2px; display: inline-block;">-4</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">6</div> <div style="border: 1px solid blue; padding: 2px; display: inline-block;">-1</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">7</div>

Simulation of Example 1

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$\boxed{1} \forall xy. \neg rxy \vee ryy$ $\boxed{2} \exists xy. \neg ryy \wedge rxy$ $\boxed{3} \exists y. \neg ryy \wedge rxy$ $\boxed{4} \neg ryy \wedge rxy$ $\boxed{-5} \neg ryy$ $\boxed{6} rxy$	$\boxed{7} \forall y. \neg rxy \vee ryy$ $\boxed{8} \forall y'. \neg ryy' \vee ry'y'$	x y		$\boxed{1}$ $\boxed{2}$ $\boxed{-2} \sqcup \boxed{3}$ $\boxed{-3} \sqcup \boxed{4}$ $\boxed{-4} \sqcup \boxed{-5}$ $\boxed{-4} \sqcup \boxed{6}$ $\boxed{-1} \sqcup \boxed{7}$ $\boxed{-1} \sqcup \boxed{8}$

Weakly Satisfiable: $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{-5}, \boxed{6}, \boxed{7}, \boxed{8}$

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<div style="border: 1px solid blue; padding: 2px; display: inline-block;">2</div> $\exists xy. \neg ryy \wedge rxy$	<div style="border: 1px solid blue; padding: 2px; display: inline-block;">9</div> $\neg rxx \vee rxx$	y		<div style="border: 1px solid blue; padding: 2px; display: inline-block;">2</div>
<div style="border: 1px solid blue; padding: 2px; display: inline-block;">3</div> $\exists y. \neg ryy \wedge rxy$	<div style="border: 1px solid blue; padding: 2px; display: inline-block;">10</div> $\neg rxy \vee ryy$			<div style="border: 1px solid blue; padding: 2px; display: inline-block;">-2</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">3</div>
<div style="border: 1px solid blue; padding: 2px; display: inline-block;">4</div> $\neg ryy \wedge rxy$				<div style="border: 1px solid blue; padding: 2px; display: inline-block;">-3</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">4</div>
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<div style="border: 1px solid blue; padding: 2px; display: inline-block;">6</div> rxy				<div style="border: 1px solid blue; padding: 2px; display: inline-block;">-4</div> \sqcup <div style="border: 1px solid blue; padding: 2px; display: inline-block;">6</div>
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Simulation of Example 1

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1 $\forall xy. \neg rxy \vee ryy$	8 $\forall y'. \neg ryy' \vee ry'y'$	x		1
2 $\exists xy. \neg ryy \wedge rxy$	9 $\neg rxx \vee rxx$	y		2
3 $\exists y. \neg ryy \wedge rxy$	-6 $\neg rxy$			-2 \sqcup 3
4 $\neg ryy \wedge rxy$	5 ryy			-3 \sqcup 4
-5 $\neg ryy$				-4 \sqcup -5
6 rxy				-4 \sqcup 6
7 $\forall y. \neg rxy \vee ryy$				-1 \sqcup 7
10 $\neg rxy \vee ryy$				-1 \sqcup 8
				-7 \sqcup 9
				-7 \sqcup 10
				-10 \sqcup 5 \sqcup -6

Strongly Unsatisfiable

States

A **state** Σ is given by

- a finite set of **passive formulas**
- a finite set of **active formulas**
- a finite set of **passive variables**
- a finite set of **active variables**
- a finite set of **clauses** \mathcal{C}_Σ

satisfying some conditions.

Terminology for States

- Σ -formula: passive or active formula
- Σ -variable: passive or active variable
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- Σ -branch: Open branch B of Σ -literals where $B \cap C \neq \emptyset$ for every Σ -clause C
- Strong Unsatisfiability = No Σ -branches

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- If rx and $\neg rwz$ are passive, then x, y, w, z are Σ -variables.
- If $s \vee t$ is passive, then s and t are Σ -formulas and

$$\overline{s \vee t} \sqcup s \sqcup t$$

is a Σ -clause.

Invariants of States

- If $s \wedge t$ is passive, then s and t are Σ -formulas and both

$$\overline{s \wedge t} \sqcup s$$

and

$$\overline{s \wedge t} \sqcup t$$

are Σ -clauses.

Invariants of States

- If $\forall x.s$ is passive and y is passive, then s_y^x is a Σ -formula and

$$\overline{\forall x.s} \sqcup s_y^x$$

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- If $\forall x.s$ is passive and y is passive, then s_y^x is a Σ -formula and

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is a Σ -clause.

- If $\exists x.s$ is passive, then there is some y such that s_y^x is a Σ -formula and

$$\overline{\exists x.s} \sqcup s_y^x$$

is a Σ -clause.

Initial State

Let A be a branch. Initial State Σ_A :

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- A - set of active formulas.

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- A - set of active formulas.
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- No passive formulas, no passive or active variables.
- Note: Exactly 1 Σ_A -branch: A
(assuming A is open)

Successor States

$$\Sigma \rightarrow \Sigma'$$

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- Every active Σ -formula is a Σ' -formula.
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- Every active Σ -variable is a Σ' -variable.
- If $\mathcal{C}_{\Sigma'}$ is unsatisfiable, then \mathcal{C}_{Σ} is unsatisfiable.

Soundness

Soundness Theorem 1: If $\Sigma \rightarrow \Sigma'$ and Σ' is strongly unsatisfiable, then every Σ -branch B is unsatisfiable.

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Soundness Theorem 2: If $\Sigma_A \rightarrow \Sigma'$ and Σ' is strongly unsatisfiable, then A is unsatisfiable.

Would like a “constructive” version:

If $\Sigma \rightarrow \Sigma'$ and Σ' is strongly unsatisfiable, then for every Σ -branch B we can construct a tableau refutation.

Completeness Outline

Start with [abstract consistency](#) and a [model existence theorem](#).

Assume we have a [path](#) $\bar{\Sigma}$ of states

$$\Sigma_0 \rightarrow \Sigma_1 \rightarrow \cdots \rightarrow \Sigma_n \rightarrow \cdots$$

Assume this path is [fair](#).

Use this to define an abstract consistency class.

Fair Paths

$\Sigma_0 \rightarrow \Sigma_1 \rightarrow \cdots \rightarrow \Sigma_n \rightarrow \cdots$ is **fair** if:

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If x is an active Σ_i -variable, then there is some $j \geq i$ such that x is a passive Σ_j -variable.

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- Every active variable eventually becomes passive.
If x is an active Σ_i -variable, then there is some $j \geq i$ such that x is a passive Σ_j -variable.
- There is eventually a passive variable.

Consistency

Assume a fair path: $\Sigma_0 \rightarrow \Sigma_1 \rightarrow \dots \rightarrow \Sigma_n \rightarrow \dots$

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 - ② there is some Σ_i -branch B such that $A \subseteq B$.
- A branch A is *i -consistent* if A is j -supported for all $j \geq i$.
- $\{A \mid \exists i. A \text{ is } i\text{-consistent}\}$ is an abstract consistency class.

Completeness

Completeness: Let A be a branch. If there is a fair path

$$\Sigma_A \rightarrow \Sigma_1 \rightarrow \Sigma_2 \rightarrow \dots$$

then A is satisfiable.

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$$\mathcal{T}_{\text{MAT}} \frac{rs_1 \dots s_n, \neg rt_1 \dots t_n}{s_1 \neq t_1 \mid \dots \mid s_n \neq t_n} \quad n \geq 1$$

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- Mating generates clauses of the form

$$\neg rs_1 \dots s_n \sqcup \neg rt_1 \dots t_n \sqcup s_1 \neq t_1 \sqcup \dots \sqcup s_n \neq t_n$$

Future Work

- Algorithm for giving tableau refutations and Coq proof terms.
- Use 1 MiniSat image iteratively.
- Make use of assignments/ Σ -branches found by MiniSat to guide the search.

Conclusion

- Relatively easy to simulate a complete ground tableau calculus by a sequence of SAT problems.

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Conclusion

- Relatively easy to simulate a complete ground tableau calculus by a sequence of SAT problems.
- Completeness via abstract consistency
- Questions? Comments? Has anyone seen something similar?

References:

- Chad E. Brown and Gert Smolka. Analytic Tableaux for Simple Type Theory and its First-Order Fragment. Logical Methods in Computer Science. Volume 6, Issue 2. 2009
- Julian Backes and Chad E. Brown. Analytic Tableaux for Higher-Order Logic with Choice. IJCAR 2010.