Models for Intuitionistic Propositional Logic

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Abstract

This report discusses two models for intuitionistic propositional logic - Heyting algebras and Kripke structures - and their properties. The relation between Kripke structures and Heyting algebras, and their relation with the natural deduction system and the Fitting's tableau systems are also discussed.

1 Heyting algebras

Definition 1 (Heyting algebra)**.** A *Heyting algebra* is a *preorder* $(H, \leq)$ with a smallest element $\bot$ and a largest element $\top$ and three operations $\wedge$, $\vee$, and $\rightarrow$ satisfying the following conditions for all $x, y, z \in H$:

(i) $x \leq \top$

(ii) $\bot \leq x$

(iii) $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$

(iv) $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$

(v) $z \leq x \rightarrow y$ iff $z \wedge x \leq y$.

We write $x \cong y$ if $x \leq y$ and $y \leq x$.

Fact 2. For any Heyting algebra H and $x, y \in H$, we have the following facts:

(i) $x \wedge y \leq x$ and $x \wedge y \leq y$

(ii) $x \wedge y \cong y \wedge x$

(iii) $x \leq x \vee y$ and $y \leq x \vee y$

(iv) $x \vee y \cong y \vee x$

(v) $(x \rightarrow y) \wedge x \leq y$

(vi) $x \leq y$ iff $x \vee y \cong y$.

Definition 3 (Valuation on a Heyting algebra)**.** A valuation of a Heyting algebra H is a function $V : P \mapsto H$ that assigns to each propositional variable a specific element of the algebra, where P is the infinite set of propositional variables. The valuation is extended to formulas recursively:

$$
\begin{aligned}
V(\bot) &= \bot \\
V(s \to t) &= V(s) \to V(t) \\
V(s \land t) &= V(s) \land V(t) \\
V(s \lor t) &= V(s) \lor V(t)
\end{aligned}
$$

Note that $\land$, $\lor$, $\to$, and $\bot$ on the left-hand side are the connectives and Falsehood constant of the logic, while on the right-hand side are the operations and smallest element of the Heyting algebra, respectively.

A valuation of a list of formulas $V(\Gamma)$ is defined as the valuation of the conjunction of the formulas in that list:

$$
\begin{aligned}
V(nil) &= \top \\
V(s, \Gamma) &= V(s) \land V(\Gamma)
\end{aligned}
$$

Definition 4 (Heyting entailment)**.** We say that s is H-entailed by Γ, denoted $\Gamma \vDash_H s$, if $V(\Gamma) \leq V(s)$ for *all* valuations V of H.

Lemma 5 (Soundness for **Ni**)**.** *Given H an arbitrary Heyting algebra, if $\Gamma \vdash^i s$, then $\Gamma \vDash_H s$.*

Proof. By induction on the derivation $\Gamma \vdash^i s$. ☐

Collorary 6 (Semantics Equivalence)**.** If $\vdash^i s \leftrightarrow t$, then for any Heyting algebra H and its valuation V, $V(s) \cong V(t)$.

Proof. By the soundness lemma we have $\vDash_H s \leftrightarrow t$, therefore $V(nil) = \top \leq V(s \leftrightarrow t) = V(s \to t) \land V(t \to s)$. Then we have $\top \leq V(s \to t)$ and $\top \leq V(t \to s)$, which lead to $V(s) \leq V(t)$ and $V(t) \leq V(s)$. ☐

Lemma 7. *If $\top \leq V(s)$ for any Heyting algebra H and any valuation V of H, then $\vdash^i s$.*

Proof. We construct a Heyting algebra H_s of *formulas* where $s \leq t = \vdash^i s \to t$. The bottom element $\bot$ and operations $\land$, $\lor$, $\to$ of H_s are respectively the constant $\bot$ and connectives $\land$, $\lor$, $\to$ of formulas. It is easy to see that H_s is a Heyting algebra. We use the valuation $V(x) = x$. By induction $V(s) = s$. Now, if $\top \leq V(s)$, then $\top \leq s$, which means that $\vdash^i \top \to s$. Therefore $\vdash^i s$. ☐

Lemma 8 (Completeness for **Ni**)**.** *If $\Gamma \vDash_H s$ for any Heyting algebra H, then $\Gamma \vdash^i s$.*

Proof. Follows from Lemma 7. ☐

Remark 1. In the original definition of Heyting algebra, the order is a partial order while here we use a preorder, since the soundness proof does not need antisymmetry. This is also observed by Brown [2014]. If antisymmetry is accepted, then we can replace $V(s) \cong V(t)$ with $V(s) = V(t)$.

Remark 2. Here we have only formalized the completeness proof for preordered Heyting algebras basing on Troelstra and Dalen [1988], in which a stronger proof for partial-ordered Heyting algebras is provided. The authors also observe that completeness holds for finite Heyting algebras.

2 Kripke structures

Definition 9 (Kripke model)**.** A Kripke model is a tuple $(K, \leq, \alpha)$ where $\leq$ is a preorder on the set of states K, and $\alpha : P \mapsto \mathcal{P}K$ is a monotonic mapping from propositional variables to subsets of K, where monotonicity means that if $p \in \alpha(x)$ and $p \leq q$ then $q \in \alpha(x)$.

Definition 10 (Valuation on a Kripke model)**.** The valuation of a formula s on a Kripke model $(K, \leq, \alpha)$ is defined recursively as:

$$\begin{aligned} \hat{K}x &:= \alpha(x) \\ \hat{K}\bot &:= \emptyset \\ \hat{K}(s \wedge t) &:= \hat{K}s \cap \hat{K}t \\ \hat{K}(s \vee t) &:= \hat{K}s \cup \hat{K}t \\ \hat{K}(s \rightarrow t) &:= \{p \in K \mid (p \uparrow) \cap \hat{K}s \subseteq \hat{K}t\} \end{aligned}$$

where $(p \uparrow) := \{q \in K \mid p \leq q\}$.

The valuation for list of formulas is defined as:

$$\begin{aligned} \hat{K}\emptyset &:= K \\ \hat{K}(s, \Gamma) &:= \hat{K}s \cap \hat{K}(\Gamma) \end{aligned}$$

Fact 11 ($\hat{K}$ is monotonic)**.** If $p \in \hat{K}s$ and $p \leq q$ then $q \in \hat{K}s$.

Proof. By induction on s. □

Definition 12 (Forcing relation)**.** The forcing relation $\vDash$ between states of a Kripke model $(K, \leq, \alpha)$ and formulas is defined as:

(i) $p \vDash x$ iff $p \in \alpha(x)$

(ii) $p \vDash \bot$ never holds

(iii) $p \vDash s \wedge t$ iff $p \vDash s$ *and* $p \vDash t$

(iv) $p \vDash s \vee t$ iff $p \vDash s$ *or* $p \vDash t$

(v) $p \vDash s \to t$ iff $\forall q \geq p,\ q \vDash s \longrightarrow q \vDash t$.

We say p *forces* s or p *satisfies* s to mean $p \vDash s$. We write $[s]_K := \{p \in K \mid p \vDash s\}$.

Fact 13. $[s]_K = \hat{K}s$.

Proof. By induction on s. ☐

Lemma 14 (Soundness for **Ni**)**.** *If* $\Gamma \vdash^i s$ *then* $\hat{K}(\Gamma) \subseteq \hat{K}s$*.*

Proof. By induction on the derivation $\Gamma \vdash^i s$. Fact 11 and the properties of the preorder are needed. ☐

Definition 15 (Upward-closed sets)**.** A set A is called upward-closed, if for any $p \in A$, $(p \uparrow) \subseteq A$. We write $\overline{K} := \{A \mid A \subseteq K \land A \text{ is upward-closed}\}$.

Fact 16. Upward-closed sets are closed under intersection and union. That is, if A and B are in $\overline{K}$, then so are $A \cap B$ and $A \cup B$.

Lemma 17 (Kripke models to Heyting algebras)**.** $\overline{K}$ *with set inclusion is a Heyting algebra, where* $\top = K$*,* $\bot = \emptyset$*, and the operations are:*

$$\begin{aligned} A \land B &:= A \cap B \\ A \lor B &:= A \cup B \\ A \to B &:= \{p \in K \mid (p \uparrow) \cap A \subseteq B\} \end{aligned}$$

The valuation for the algebra is $V(x) = \hat{K}x = [x]_K = \alpha(x) \in \overline{K}$*.*

Fact 18. If $\overline{K}$ and V are respectively the Heyting algebra and its valuation built from K, then $V(s) = [s]_K$.

Proof. By induction on s. ☐

Fact 19. If K is finite then $\overline{K}$ is a finite distributive lattice and we can define $A \to B := \bigvee\{C \in \overline{K} \mid C \land A \leq B\}$: the join is finite.

3 Countermodels

We demonstrate several counter-models for some propositionally underivable formulas. The counter-models are given as Kripke models $(K, \leq, \alpha)$, whose states represent α: each state p is a set of (labeled by) variables whose mapping by α contains p. The models are visualized in form of lattices of Heyting algebras. These models are from unpublished notes by Prof. Gert Smolka.

Fact 20 (Countermodel for $\bot$, x, and $x \lor y$)**.** $\nvdash^i \bot$, and $\nvdash^i x$, and $\nvdash^i x \lor y$.

Proof. Let $K := \{\emptyset\}$. We have $\overline{K} = \{\emptyset, K\}$, then $\hat{K}\bot = \hat{K}x = \hat{K}(x \lor y) = \emptyset$, that is $K = \top \nleq V(\bot) = V(x) = V(x \lor y)$.

$$\begin{array}{c} K \\ | \\ \emptyset \end{array}$$

□

Fact 21 (Countermodel for $\neg x$)**.** $\nvDash^i \neg x$.

Proof. Let $K := \{\{x\}\}$. Then $\overline{K} = \{\emptyset, K\}$. We have $\hat{K}x = K$ and $\hat{K}(\neg x) = \emptyset$.

$$\begin{array}{c} K \\ | \\ \emptyset \end{array}$$

□

Fact 22 (Countermodel for **XM** and **DN**)**.** $\nvDash^i x \vee \neg x$ and $\nvDash^i \neg\neg x \to x$.

Proof. Let $K := \{\emptyset, \{x\}\}$. Then $\overline{K} = \{\emptyset, \{\{x\}\}, K\}$. We have

$$\begin{aligned} \hat{K}x &= \{\{x\}\} \\ \hat{K}(\neg x) &= \emptyset \\ \hat{K}(x \vee \neg x) &= \hat{K}x \\ \hat{K}(\neg\neg x) &= K \\ \hat{K}(\neg\neg x \to x) &= \hat{K}x \end{aligned}$$

$$\begin{array}{c} K \\ | \\ \hat{K}x \\ | \\ \emptyset \end{array}$$

□

Fact 23 (Countermodel for **Peirce**)**.** $\nvDash^i ((x \to y) \to x) \to x$.

Proof. Let $K := \{\emptyset, \{x\}, \{x, y\}\}$. Then $\overline{K} = \{\emptyset, \{\{x, y\}\}, \{\{x\}, \{x, y\}\}, K\}$. We have

$$\begin{aligned} \hat{K}y &= \{\{x, y\}\} \\ \hat{K}x &= \{\{x\}, \{x, y\}\} \\ \hat{K}(x \to y) &= \hat{K}y \\ \hat{K}((x \to y) \to x) &= K \\ \hat{K}(((x \to y) \to x) \to x) &= \hat{K}x \end{aligned}$$

$$
\begin{tikzcd}
K \arrow[d, no head] \\
\hat{K}x \arrow[d, no head] \\
\hat{K}y \arrow[d, no head] \\
\emptyset
\end{tikzcd}
$$

$\square$

4 Demos

Definition 24 (Hintikka pairs)**.** A pair of lists of formulas (Γ, Δ) is called a Hintikka pair if it satisfies all of the following conditions:

(i) $\bot \notin \Gamma$

(ii) if $x \in \Gamma$ then $x \notin \Delta$

(iii) if $s \to t \in \Gamma$ then $s \in \Delta$ *or* $t \in \Gamma$

(iv) if $s \wedge t \in \Gamma$ then $s \in \Gamma$ *and* $t \in \Gamma$

(v) if $s \vee t \in \Gamma$ then $s \in \Gamma$ *or* $t \in \Gamma$

(vi) if $s \wedge t \in \Delta$ then $s \in \Delta$ *or* $t \in \Delta$

(vii) if $s \vee t \in \Delta$ then $s \in \Delta$ *and* $t \in \Delta$.

Definition 25 (Demos)**.** A demo, or a Hintikka collection, is a finite set $\mathcal{D}$ of pairs (Γ, Δ) such that any $(\Gamma, \Delta) \in \mathcal{D}$ is a Hintikka pair *and* if $s \to t \in \Delta$ then there exists $(\Gamma', \Delta') \in \mathcal{D}$ such that $s, \Gamma \subseteq \Gamma'$ and $t \in \Delta'$.

Definition 26 (Positive set inclusion)**.** A pair (Γ, Δ) is a positive subset of (Γ', Δ'), written $(\Gamma, \Delta) \subseteq^+ (\Gamma', \Delta')$, if $\Gamma \subseteq \Gamma'$.

Fact 27. If $\mathcal{D}$ is a finite set of pairs of lists of formulas and $\alpha(x) := \{(\Gamma, \Delta) \in \mathcal{D} \mid x \in \Gamma\}$, then $(\mathcal{D}, \subseteq^+, \alpha)$ is a finite Kripke model.

Proof. It is clear that $\subseteq^+$ is a preorder. If $(\Gamma, \Delta) \in \alpha(x)$ and $(\Gamma', \Delta') \subseteq^+ (\Gamma', \Delta')$, then $x \in \Gamma \subseteq \Gamma'$ and therefore $(\Gamma', \Delta') \in \alpha(x)$. Thus we have showed that α is monotonic. $\square$

Lemma 28. *Let $\mathcal{D}$ a demo and $(\Gamma, \Delta) \in \mathcal{D}$, and $(\mathcal{D}, \subseteq^+, \alpha)$ is the Kripke model of $\mathcal{D}$. Then:*

(i) $(\Gamma, \Delta) \in \hat{\mathcal{D}}s$ for every $s \in \Gamma$ and

(ii) $(\Gamma, \Delta) \notin \hat{\mathcal{D}}s$ for every $s \in \Delta$.

Proof. By induction on s. $\square$

Definition 29. A demo $\mathcal{D}$ **falsifies** (Γ, Δ) if $\mathcal{D}$ contains a pair (Γ', Δ') such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

Theorem 30. *If* (Γ, Δ) *is falsified by a demo, then* $\Gamma \nvdash^i \bigvee \Delta$.

Proof. We have for the demo $\mathcal{D}$ that falsifies (Γ, Δ) a pair (Γ', Δ') such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

By definition we have

$$\hat{\mathcal{D}}(\Gamma) = \bigcap\{\hat{\mathcal{D}}s \mid s \in \Gamma\}$$
$$\hat{\mathcal{D}}(\bigvee \Delta) = \bigcup\{\hat{\mathcal{D}}s \mid s \in \Delta\}$$

From Lemma 28, we have for all $s \in \Gamma \subseteq \Gamma'$, $(\Gamma', \Delta') \in \hat{\mathcal{D}}s$, and for all $s \in \Delta \subseteq \Delta'$, $(\Gamma', \Delta') \notin \hat{\mathcal{D}}s$. Therefore $(\Gamma', \Delta') \in \hat{\mathcal{D}}(\Gamma)$ and $(\Gamma', \Delta') \notin \hat{\mathcal{D}}(\bigvee \Delta)$, i.e. $\hat{\mathcal{D}}(\Gamma) \nsubseteq \hat{\mathcal{D}}(\bigvee \Delta)$. By the soundness lemma 14 we finally have $\Gamma \nvdash^i \bigvee \Delta$. $\square$

Lemma 31. $\Gamma \Rightarrow_F \Delta$ *is decidable.*

We call a pair (Γ, Δ) **consistent** if $\Gamma \not\Rightarrow_F \Delta$.

Fact 32. If (Γ, Δ) is consistent and $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, then (Γ', Δ') is also consistent.

Definition 33 (Maximal consistent extension)**.** (Γ_m, Δ_m) is a maximal consistent extension of (Γ, Δ) if (Γ_m, Δ_m) is consistent and contains only subformulas of Γ or Δ, and for any other (Γ', Δ') that satisfies the same properties, if $\Gamma_m \subseteq \Gamma'$ and $\Delta_m \subseteq \Delta'$, then $\Gamma' \subseteq \Gamma_m$ and $\Delta' \subseteq \Delta_m$.

We write $\mathcal{D}(\Gamma, \Delta)$ to denote the set of all maximal consistent extensions of (Γ, Δ).

Lemma 34. $\mathcal{D}(\Gamma, \Delta)$ *is a demo.* We call $\mathcal{D}(\Gamma, \Delta)$ the **canonical demo** of (Γ, Δ).

Lemma 35. *For any consistent* (Γ', Δ') *that contains only subformulas of* Γ *or* Δ*, there exists a pair* $(\Gamma_m, \Delta_m) \in \mathcal{D}(\Gamma, \Delta)$ *such that* $\Gamma' \subseteq \Gamma_m$ *and* $\Delta' \subseteq \Delta_m$*.*

Collorary 36. If (Γ, Δ) is consistent, then $\mathcal{D}(\Gamma, \Delta)$ falsifies (Γ, Δ).

Lemma 37. *Either* $\Gamma \Rightarrow_F \Delta$ *or* $\mathcal{D}(\Gamma, \Delta)$ *falsifies* (Γ, Δ)*.*

Lemma 38. $\Gamma \vdash^i s$ *iff* $\Gamma \Rightarrow_F s$ *and* $\Gamma \vdash^i \bigvee \Delta$ *iff* $\Gamma \Rightarrow_F \Delta$*.*

Lemma 39. *If* s *is a subformula of* (Γ, Δ)*, and* (Γ_m, Δ_m) *is a maximal consistent extension of* (Γ, Δ)*, and* $s \notin \Gamma_m \cup \Delta_m$*, then both* $((s, \Gamma_m), \Delta_m)$ *and* $(\Gamma_m, (s, \Delta_m))$ *are inconsistent.*

Collorary 40. If s is a subformula of (Γ, Δ), and (Γ_m, Δ_m) is a maximal consistent extension of (Γ, Δ), then either $s \in \Gamma_m$ or $s \in \Delta_m$.

Theorem 41 (Maximal consistent extension identity)**.** *If* (Γ_1, Δ_1) *and* (Γ_2, Δ_2) *are both maximal consistent extensions of* (Γ, Δ)*, and* Γ_1 *and* Γ_2 *have the same set of propositional variables, and* Δ_1 *and* Δ_2 *have the same set of implications, then* $(\Gamma_1, \Delta_1) \equiv (\Gamma_2, \Delta_2)$*, i.e.* $\Gamma_1 \equiv \Gamma_2$ *and* $\Delta_1 \equiv \Delta_2$*.*

Lemma 42. *If* (Γ_1, Δ_1) *and* (Γ_2, Δ_2) *are both maximal consistent extensions of* (Γ, Δ)*, and* $\Gamma_1 \equiv \Gamma_2$*, then* $\Delta_1 \equiv \Delta_2$*.*

Collorary 43. $(\mathcal{D}(\Gamma, \Delta), \subseteq^+)$ is a partial order.

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