Research Immersion Lab Summary

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Abstract

This is the summary of the research immersion lab on propositional logics, supervised by Prof. Dr. Gert Smolka. This report only discusses current results and problems at the abstract level, including those important ideas provided by Prof. Smolka. For proof and technical details and references please refer to the accompanying documents and Coq source code.

Objectives and Results

The project aims at studying properties of classical and, more importantly, intuitionisitc propositional logics, in both proof-theoretic and model-based approaches. We study systems for the two logics, including natural deduction systems, Gentzen's sequent calculi, and Hilbert's axiomatic systems. In the model-based direction, we study boolean semantics for classical propositional logic, and Kripke models and Heyting algebras for intuitionistic propositional logic. Tableau methods are also studied for both logics, to prove their decidability. Especially the Fitting tableau system for intuitionistic logic provides insights to understand the logic's semantics clearly.

More specifically, we have the following results formalized constructively in Coq:

1. We formulate natural deduction systems, Gentzen's sequent calculi, and Hilbert's systems for both logics. We prove the equivalence between the three systems, separately for each logic.

2. We prove the important Cut elimination for Gentzen's both classical and intuitionistic systems using pure structural inductions, avoiding the formulation of derivation trees and their depths. The systems are based on Kleene's variants.

3. We prove the decidability of both classical and intuitionistic propositional logics proof-theoretically, using the idea of terminating proof search procedure on Gentzen's systems.

4. We prove the equivalence between classical propositional logic, represented by either the natural deduction system or the Gentzen system, and boolean entailment. And by the tableau method for boolean semantics we also have another decidability proof for the classical logic.

5. We prove the equivalence between intuitionisitc propositional logic, represented by the Gentzen system, and the Fitting's tableau system. By looking at the set of all derivable "clauses" constrained by the subformula property, we prove the decidability of the tableau system, therefore obtain another decidability proof for the intuitionistic logic.

6. We formulate Heyting algebras as a semantic model for intuitionistic logic. We prove soundness and completeness of preordered Heyting algebras for the logic. The proofs are simpler — for completeness of partial-ordered Heyting algebras we have to work with quotients — and therefore weaker than known results in the literature.

7. We also formulate Kripke models as another semantic model for intuitionistic logic. We only prove soundness. We also prove the construction from Kripke models to Heyting algebras.

8. We provide constructive and therefore computational (although not efficient enough) proofs to evaluate **truth values** of intuitionistic formulas in finite Heyting algebras. We also provide procedure to evaluate formulas on finite Kripke models. This allows us to have compact proofs for known underivable results, and more importantly the independence of intuitionistic connectives, of which we show both McKinsey's countermodels and our simpler countermodels.

9. Kripke models and Heyting algebras show the connection between the two logics in the aspect of truth values.

10. Fitting's tableau method can produce countermodels for underivable formulas in form of Kripke models. By apply the tableau rules, we can obtain a set of maximal underivable subformula clauses, which we call the **canonical demo**. We show how the canonical demo is a Kripke model and how it shows underivability.

11. We show that the canonical demo is actually a partial order, under the positive formula subset relation. We also show that the canonical demo can be reduced to a compact representation of clauses of only positive variables and negative implications. Our effort to prove the stronger statement that uses only positive variables failed — in fact counterexamples were found.

12. By looking at Kripke models and Fitting's tableau rules, we see that terminal nodes in a Kripke model behave classically, and it is the non-terminal nodes and the connections are what create the intuitionistic sense. Since the non-terminal nodes can only be produced by negative implications in the tableau rules, a simple result was found that implication free formulas are equivalently classically and intuitionistically derivable. One might wish to obtain a seemingly stronger result: *if the set of subformulas generated by the tableau rules does not contain negative implications*, then the formula is equivalently classically and intuitionistically derivable. This can be proved, but unfortunately, these 2 facts are equivalent: due to the tableau rules, the only way to go from negative formulas to positive formulas is the negative implication rule; and we always start with a negative formula.

Discussions and Problems

1. We want to know how to show cut-elimination for boolean entailment.

   Also, does our cut elimination follow Gentzen's original proof?

   Proofs by Troelstra and Schwitchtenberg make explicit use of the derivation trees and their depth and size, which allows them to state lemmas with depth contraints on the derivation trees.

2. Currently we already have a computable informative decision procedure: $\{\Gamma \vdash^i s\} + \{K \mid \hat{K}(\Gamma) \nsubseteq \hat{K}s\}$, where K is actually the canonical demo. Since we can construct a Heyting algebra from a Kripke model, we can also have the decision procedure $\{\Gamma \vdash^i s\} + \{(H, V) \mid V(\Gamma) \nleq V(s)\}$.

   We want efficient automated proof for intuitionistic formula (proof by reflection). We have computable evaluation procedures on Kripke models and Heyting algebras. The construction of the canonical demo using tableau rules depends on the decision procedure of tableau, which is very inefficient, and therefore should not be used. The tableau decision procedure, however, search for set of derivable clauses in the subformula set, so the complement of such set is the set of consistent clauses. A simple filter can be applied to get the maximal consistent clauses. This is faster than the naive tableau construction, but still we have to work with a set with size exponential on the formula's size. Nevertheless, we have the idea for a more economical way to find the compact representation of the canonical demo using the maximal consistent extension identity lemma.

   The evaluation on Kripke models is acceptable. The evaluation on Heyting algebras relies on the Kripke-Heyting translation, but is not mandantory — Kripke models are enough for our purpose. The model finding procedure's complexity is, at best, exponential on the number of variables and implications in the formula.

3. For classical underivable formulas, the counter Kripke model **should** only have 1 state, which means that the corresponding counter Heyting algebra **should** have exactly 2 truth values $\top$ and $\bot$ and the formula evaluate to $\bot$. For classical derivable but intuitionisitc underivable formulas, the Heyting algebra **should** have at least 3 truth values, and the Kripke model **should** have at least 1 non-terminal node.

   The more general and important question is, **given a formula, can we identify its minimal countermodel, or at least the size of its minimal countermodel?** The result can reduce the size of the search space for countermodels.

4. We were optimistic that, it is enough to look for countermodels from the set of subsets of the powerset of the formula's variables. This, however, was disproved by a few computer-found counterexamples:

   - $\neg x \vee \neg\neg x$
   - $\neg x \vee \neg x \to x$
   - $\neg x \vee \neg x \to y$
   - $x \to y \vee \neg x \to y$
   - $x \to y \vee (x \to y) \to y$

Nevertheless, we have the maximal consistent extension identity lemma, which suggests that a canonical demo has a compact representation as a set of sets of positive variables and negative implications, which means that our search space is the set of subsets of powerset of the formula's variables and implications, more specifically positive variables and negative implications.

The subformulas generated by the tableau rules should narrow down the search space even more.

5. The counterexample for the variable-only search idea suggests some property of the formula that makes the variable-only search fail. We do not know yet what this property is, but it does not seem to be simple or local, like the Harrop property. The property possibly ranges over multiple levels of the formula's tree representation.

6. A much simpler property is the implication-free property discussed in the previous section. The proof for the implication-free property depends on the classical Gentzen system and the tableau system. Can this suggest some relation between the two systems? Is there a similar proof for the natural deduction system?

7. What is the connection between the Kleene system and the Fitting system?

8. Can we construct demo from the Kleene system? The current answer is probably NO.

   The other question is, can we obtain decidability by Heyting/Kripke demo from the Kleene system?

9. Can we have a similar cut-elim proof for **G3ip** where the left implication rule is

$$\frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow s \qquad \Gamma \setminus s \to t, t \Rightarrow u}{\Gamma \Rightarrow u}$$

   The current answer is probably NO.

10. Technically, we want to set up rewriting in Coq for semantically equivalent formulas.