The story so far	Refining Corecursion	Application	Beyond this talk

# Formalizing Stream-Calculus in Coq Corecursion Revised

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The story so far	Refining Corecursion	Application	Beyond this talk

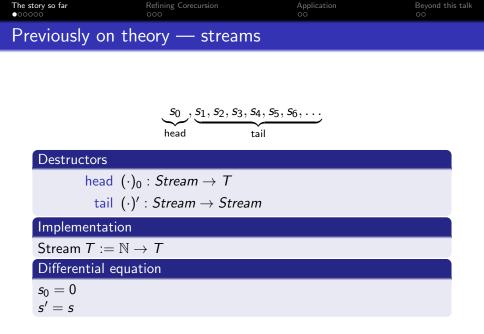
#### 1 The story so far

- Streams
- Equality
- Causality
- Ring

# 2 Refining Corecursion

- 3 Application
  - Squareroot
  - Catalan numbers

# 4 Beyond this talk



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 The story so far
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 Previously on theory — equality
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#### Equality

$$\begin{array}{l} (a=b):=a_0=b_0\wedge a'=b' \quad (\leftrightarrow \forall n,a_n=b_n) \\ (a=_nb):=a_0=b_0\wedge a'=_{n-1}b' \quad (\leftrightarrow \forall n'< n,a_{n'}=b_{n'}) \end{array}$$

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Previously on th	eory — causality		

# Example differential equations

$$w_0 := 0 w' := w' + 1$$

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Previously on th	eory — causality		

#### Example differential equations

 1  $u_0 := 0$  u' := u  $tc_u(s) = s$  2  $v_0 := 0$  v' := v'' 1  $tc_v(s) = s''$  3  $w_0 := 0$  w' := w' + 1 $tc_w(s) = s' + 1$ 

Tail characterization of single stream

 $tc: Stream \rightarrow Stream$ tc s = s'

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Previous	ly on theory –	— causality		
Examp	ole differential equ	ations		
<b>1</b> u	0 := 0			
и	':=u	$tc_u(s) = s$		
	$_{0} := 0$			
V	' := v''	$tc_{v}(s)=s''$		
	$v_0 := 0$			
и	v' := w' + 1	$tc_w(s) = s' + 1$		
Tail cl	naracterization of	single stream		
	ream  ightarrow Stream			
<i>tc s</i> =	-			
Causa	lity			

 $\forall a_1 =_n a_2 \rightarrow tc \ a_1 =_n tc \ a_2$ 

Previously on in	nnlementation —	- corec	
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Tail characterization of operations

$$tc: X \to (X \to Stream) \to Stream$$
  
 $tc \times o = (o \times)'$ 



The story so far ○○○●○○	Refining Corecursion	OO	OO
Previously on	implementation –	– corec	

Tail characterization of operations

$$tc: X \rightarrow (X \rightarrow Stream) \rightarrow Stream$$
  
 $tc \times o = (o \times)'$ 

#### Causality

# $\forall n, i : \forall a_1, a_2, (\forall x, a_1 x =_n a_2 x) \rightarrow tc i a_1 =_n tc i a_2$

The s	tory so far 00	Refining Corecursion	Application 00	Beyond this talk 00
Pr	eviously on im	plementation -	— corec	
	Tail characteriza	tion of operations		
	· · ·	Stream) $ ightarrow$ Stream	ו	
	tc x o = (o x)'			

### Causality

$$\forall n, i : \forall a_1, a_2, (\forall x, a_1 x =_n a_2 x) \rightarrow tc i a_1 =_n tc i a_2$$

### Corecursion

$$h: (X o T)$$
  
 $tc: X o (X o Stream) o Stream$   
 $corec \ h \ tc: X o Stream$ 

#### tc causal

 $(corec h tc x)_0 = h x$ (corec h tc x)' = tc x (corec h tc)

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The story so far ○○○○●○	Refining Corecursion	Application 00	Beyond this talk
Previously on th	eory — special stre	ams	

Streams	
0 <sub>0</sub> := 0	$0 = (0, 0, 0, 0, 0, \dots)$
0' := 0	
$[t]_0 := t$	[t] = (t, 0, 0, 0, 0,)
[t]' := 0	
1 := [1]	1 = (1, 0, 0, 0, 0,)
$X_0 := 0$	$X = (0, 1, 0, 0, 0, \ldots)$
X' := 1	

The story so far ○○○○○● Refining Corecursion

Application 00 Beyond this talk

# Previously on theory — ring

#### Addition

$$(a+b)_0 := a_0 \bigoplus b_0 (a+b)' := (a') + (b')$$

### Subtraction

$$(-a)_0 := -(a_0) (-a)' := -(a')$$

#### Multiplication

#### Division

$$(s^{-1})_0 := (s_0)^{-1} \ (s^{-1})' := -s' \times ([s_0] \times s)^{-1}$$

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The story so	Far Refining Corecursion ●00	Application 00	Beyond this talk 00
Corec	ursion with general input		
Тур			
tc	$X \to T$ $X \to (X \to Stream) \to Stream$ $X \to C \to Stream$ $X \to Stream$	1	
tc	causal		
•	corec converges		
•	$(\operatorname{corec} h tc x)' = tc x (\operatorname{corec} h tc)$	)	
Car	isality		
	$\forall i:$ $\forall a_1, a_2: (\forall x: a_1 x)$	- 20 X) ->	
	va1, a2 . (vx . a1 x	$-n a_2 \wedge j \rightarrow$	

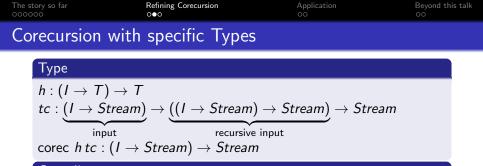
 $tc i a_1 =_n tc i a_2$ 

The story so far 000000	Refining Corecursion ○●○	Application 00	Beyond this talk 00
Corecursion w	ith specific Types		
Туре			
$h: (I \rightarrow T) -$	$\rightarrow T$ am) $\rightarrow ((1 \rightarrow \text{Stream})$	$\rightarrow$ Stream) $\rightarrow$ Stream	aam

recursive input

input

corec h tc :  $(I \rightarrow Stream) \rightarrow Stream$ 



#### Causality

$$\forall i_1, i_2 : (\forall y : i_1 y =_{(n+1)} i_2 y) \rightarrow$$
  
 
$$\forall a_1, a_2 : (\forall x_1, x_2 : (\forall y : x_1 y =_n x_2 y) \rightarrow a_1 x_1 =_n a_2 x_2) \rightarrow$$
  
 
$$tc i_1 a_1 =_n tc i_2 a_2$$

The st	tory so far 00	Refining Corecursion ○●○	Application 00	Beyond this talk		
Со	Corecursion with specific Types					
	Туре					
	$h:(I \rightarrow T) \rightarrow T$	-				
	$tc: (I \rightarrow Stream)$	$(I \rightarrow Streak)$	$m) \rightarrow Stream) \rightarrow Stream$			
	input		ive input			
	corec $h tc : (I \rightarrow$	$Stream) \rightarrow Stream$	am			
	Causality					

$$\forall i_1, i_2 : (\forall y : i_1 y =_{(n+1)} i_2 y) \rightarrow$$
  
 
$$\forall a_1, a_2 : (\forall x_1, x_2 : (\forall y : x_1 y =_n x_2 y) \rightarrow a_1 x_1 =_n a_2 x_2) \rightarrow$$
  
 
$$tc i_1 a_1 =_n tc i_2 a_2$$

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#### tc causal

- corec converges
- (corec h tc x)' = tc x (corec h tc)
- corec *h* tc causal

The story so far	Refining Corecursion	Application	Beyond this talk
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Framework -	– Summarv		

- corec h tc
- tc causal
  - $(\operatorname{corec} h tc x)' = tc x (\operatorname{corec} h tc)$
  - (corec *h tc*) causal
- $=_n$ -Rewriting and Properness
- Ring tactic familiy

The story so far 000000	Refining Corecursion	Application ●○	Beyond this talk
Squareroot			

#### Characterization

$$\sqrt{s} \times \sqrt{s} = s$$
$$(\sqrt{s})_0 = \sqrt{(s_0)} =: h$$
$$(\sqrt{s})' = \frac{s'}{[\sqrt{s_0}] + \sqrt{s}} =: t$$

# Corecursive definition

$$\sqrt{s} = corec \ h \ t \ s$$

The story so far	Refining Corecursion	Application	Beyond this talk
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Catalan numbers			

$$C_0 = 0$$
  

$$C_1 = 1$$
  

$$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^n C_{k+1} \cdot C_{n-k+1}$$

Catalan numbers	5		
The story so far	Refining Corecursion	Application	Beyond this talk
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$$C_0 = 0$$
  

$$C_1 = 1$$
  

$$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^n C_{k+1} \cdot C_{n-k+1}$$

# Definition as stream

$$cat_0 = 0$$
  
 $cat_1 = 1 =: h$   
 $cat'' = cat' \times cat' =: t$ 

The story so far	Refining Corecursion	Application	Beyond this talk
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Catalan numbers			

$$C_0 = 0$$
  

$$C_1 = 1$$
  

$$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^n C_{k+1} \cdot C_{n-k+1}$$

# Definition as stream

$$cat_0 = 0$$
  
 $cat_1 = 1 =: h$   
 $cat'' = cat' \times cat' =: t$   
 $cat = corec h t$ 

The story so far	OOO	Application ○●	00
Catalan num	nbers		

$$C_0 = 0$$
  

$$C_1 = 1$$
  

$$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^n C_{k+1} \cdot C_{n-k+1}$$

t

# Definition as stream

$$cat_0 = 0$$
  
 $cat_1 = 1 =: h$   
 $cat'' = cat' \times cat' =:$   
 $cat = corec h t$ 

### Closed formular

$$cat = X + cat imes cat$$
  
 $cat = rac{1 - \sqrt{1 - 4X}}{2}$ 

The story so far	Refining Corecursion	Application	Beyond this talk
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# Next steps and further work

- Shorter proofs: AC-rewriting and more automation
- Formalize ring based on shuffle product (multiplication wrt. exponential generating functions) using corecursion
- Make simultaneous use of both ring structures (convolution and shuffle product).

The story so far	Refining Corecursion	Application	Beyond this talk
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Further Rea	ding		

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   A coinductive calculus of streams.
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# Division revised

# Characterization

$$(1 + X \times s) \times (1 + X \times s)^{-1} = 1$$
  

$$(1 + X \times s)_0^{-1} = 1$$
  

$$(1 + X \times s)_1^{-1} = -s_0 =: h$$
  

$$((1 + X \times s)^{-1})'' = -\frac{s}{((1 + X \times s)^{-1})'} - s' =: t$$

#### Corecursive definition

$$((1 + X \times s)^{-1})' = \text{corec } h t s$$
  
(1 + X \times s)^{-1} = 1 + X \times ((1 + X \times s)^{-1})'  
v^{-1} = ([v\_0^{-1}] \times v)^{-1} \times [v\_0^{-1}]

# Squareroot

# Characterization

$$\sqrt{1 + X \times s} \times \sqrt{1 + X \times s} = 1 + X \times s$$
$$(\sqrt{1 + X \times s})_0 = 1$$
$$(\sqrt{1 + X \times s})_1 = \frac{s_0}{2} =: h$$
$$(\sqrt{1 + X \times s})'' = \frac{s'}{2} - \frac{s \times \sqrt{1 + X \times s'}}{4 + 2X \times \sqrt{1 + X \times s'}} =: t$$

# Corecursive definition

$$\sqrt{1 + X \times s}' = \text{corec } h \, t \, s$$
$$\sqrt{1 + X \times s} = 1 + X \times \sqrt{1 + X \times s}'$$
$$\sqrt{v} = \sqrt{[v_0^{-1}] \times v} \times [\sqrt{v_0}]$$

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