Formalizing Stream-Calculus in Coq

Corecursion Revised

Daniel Freiermuth

Advisor: Steven Schäfer
Universität des Saarlandes

July 20, 2018

Previously on theory — streams

$$\underbrace{s_0}_{\text{head}}, \underbrace{s_1, s_2, s_3, s_4, s_5, s_6, \ldots}_{\text{tail}}$$

Destructors

$$\text{head} \quad (\cdot)_0 : \mathit{Stream} \to T$$
$$\text{tail} \quad (\cdot)' : \mathit{Stream} \to \mathit{Stream}$$

Implementation

Stream $T := \mathbb{N} \to T$

Differential equation

$s_0 = 0$
$s' = s$

Previously on theory — equality

Equality

$$(a = b) := a_0 = b_0 \wedge a' = b' \qquad (\leftrightarrow \forall n, a_n = b_n)$$
$$(a =_n b) := a_0 = b_0 \wedge a' =_{n-1} b' \qquad (\leftrightarrow \forall n' < n, a_{n'} = b_{n'})$$

Previously on theory — causality

Example differential equations

1. $u_0 := 0$
   $u' := u$
2. $v_0 := 0$
   $v' := v''$
3. $w_0 := 0$
   $w' := w' + 1$

Previously on theory — causality

Example differential equations

1. $u_0 := 0$
   $u' := u \qquad tc_u(s) = s$
2. $v_0 := 0$
   $v' := v'' \qquad tc_v(s) = s''$
3. $w_0 := 0$
   $w' := w' + 1 \qquad tc_w(s) = s' + 1$

Tail characterization of single stream

$tc : Stream \to Stream$
$tc\, s = s'$

Previously on theory — causality

Example differential equations

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3. $w_0 := 0$
   $w' := w' + 1$ $\qquad tc_w(s) = s' + 1$

Tail characterization of single stream

$tc : Stream \to Stream$
$tc\ s = s'$

Causality

$\forall a_1 =_n a_2 \to tc\ a_1 =_n tc\ a_2$

Previously on implementation — corec

Tail characterization of operations

$tc : X \to (X \to Stream) \to Stream$

$tc\, x\, o = (o\, x)'$

Previously on implementation — corec

Tail characterization of operations

$tc : X \to (X \to Stream) \to Stream$
$tc\, x\, o = (o\, x)'$

Causality

$\forall n, i : \forall a_1, a_2, (\forall x, a_1\, x =_n a_2\, x) \to tc\ i\ a_1 =_n tc\ i\ a_2$

Previously on implementation — corec

Tail characterization of operations

$tc : X \to (X \to Stream) \to Stream$
$tc\, x\, o = (o\, x)'$

Causality

$\forall n, i : \forall a_1, a_2, (\forall x, a_1\, x =_n a_2\, x) \to tc\, i\, a_1 =_n tc\, i\, a_2$

Corecursion

$h : (X \to T)$
$tc : X \to (X \to Stream) \to Stream$
$corec\, h\, tc : X \to Stream$

tc causal

$(corec\, h\, tc\, x)_0 = h\, x$
$(corec\, h\, tc\, x)' = tc\, x\, (corec\, h\, tc)$

Previously on theory — special streams

Streams

$$0_0 := 0 \qquad\qquad 0 = (0, 0, 0, 0, 0, \ldots)$$
$$0' := 0$$
$$[t]_0 := t \qquad\qquad [t] = (t, 0, 0, 0, 0, \ldots)$$
$$[t]' := 0$$
$$1 := [1] \qquad\qquad 1 = (1, 0, 0, 0, 0, \ldots)$$
$$X_0 := 0 \qquad\qquad X = (0, 1, 0, 0, 0, \ldots)$$
$$X' := 1$$

Previously on theory — ring

Addition

$(a+b)_0 := a_0 \bigoplus b_0$
$(a+b)' := (a') + (b')$

Subtraction

$(-a)_0 := -(a_0)$
$(-a)' := -(a')$

Multiplication

$(a \times b)_0 := a_0 \bigotimes b_0$
$(a \times b)' := a_0 \times b' + a' \times b$
$s = s_0 + X \times s'$

Division

$(s^{-1})_0 := (s_0)^{-1}$
$(s^{-1})' := -s' \times ([s_0] \times s)^{-1}$

Corecursion with general input

Type

$h : X \to T$

$tc : \underbrace{X}_{\text{input}} \to \underbrace{(X \to Stream)}_{\text{recursive input}} \to Stream$

corec $h\,tc : X \to Stream$

tc causal

- corec converges
- $(\text{corec } h\,tc\,x)' = tc\,x\,(\text{corec } h\,tc)$

Causality

$$\forall i : \\ \forall a_1, a_2 : (\forall x : a_1\,x =_n a_2\,x) \to \\ tc\,i\,a_1 =_n tc\,i\,a_2$$

Corecursion with specific Types

Type

$h : (I \to T) \to T$

$tc : \underbrace{(I \to Stream)}_{\text{input}} \to \underbrace{((I \to Stream) \to Stream)}_{\text{recursive input}} \to Stream$

$\text{corec } h\, tc : (I \to Stream) \to Stream$

Corecursion with specific Types

Type

$h : (I \to T) \to T$

$tc : \underbrace{(I \to Stream)}_{\text{input}} \to \underbrace{((I \to Stream) \to Stream)}_{\text{recursive input}} \to Stream$

$\text{corec } h\, tc : (I \to Stream) \to Stream$

Causality

$$\forall i_1, i_2 : (\forall y : i_1\, y =_{(n+1)} i_2\, y) \to$$
$$\forall a_1, a_2 : (\forall x_1, x_2 : (\forall y : x_1\, y =_n x_2\, y) \to a_1\, x_1 =_n a_2\, x_2) \to$$
$$tc\, i_1\, a_1 =_n tc\, i_2\, a_2$$

Corecursion with specific Types

Type

$$h : (I \to T) \to T$$

$$tc : \underbrace{(I \to Stream)}_{\text{input}} \to \underbrace{((I \to Stream) \to Stream)}_{\text{recursive input}} \to Stream$$

$$\text{corec } h\, tc : (I \to Stream) \to Stream$$

Causality

$$\forall i_1, i_2 : (\forall y : i_1\, y =_{(n+1)} i_2\, y) \to$$
$$\forall a_1, a_2 : (\forall x_1, x_2 : (\forall y : x_1\, y =_n x_2\, y) \to a_1\, x_1 =_n a_2\, x_2) \to$$
$$tc\; i_1\; a_1 =_n tc\; i_2\; a_2$$

tc causal

- corec converges
- $(\text{corec } h\, tc\, x)' = tc\, x\, (\text{corec } h\, tc)$
- corec $h\, tc$ causal

Framework — Summary

- corec h tc
- tc causal
  - $(\text{corec}\, h\, tc\, x)' = tc\, x\, (\text{corec}\, h\, tc)$
  - $(\text{corec}\, h\, tc)$ causal
- $=_n$-Rewriting and Properness
- Ring tactic familiy

Squareroot

Characterization

$$\sqrt{s} \times \sqrt{s} = s$$
$$(\sqrt{s})_0 = \sqrt{(s_0)} =: h$$
$$(\sqrt{s})' = \frac{s'}{[\sqrt{s_0}] + \sqrt{s}} =: t$$

Corecursive definition

$$\sqrt{s} = \mathit{corec}\ h\ t\ s$$

Catalan numbers

Definition as sequence

$C_0 = 0$
$C_1 = 1$
$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^{n} C_{k+1} \cdot C_{n-k+1}$

Catalan numbers

Definition as sequence

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$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^{n} C_{k+1} \cdot C_{n-k+1}$

Definition as stream

$cat_0 = 0$

$cat_1 = 1 =: h$

$cat'' = cat' \times cat' =: t$

Catalan numbers

Definition as sequence

$C_0 = 0$
$C_1 = 1$
$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^{n} C_{k+1} \cdot C_{n-k+1}$

Definition as stream

$cat_0 = 0$
$cat_1 = 1 =: h$
$cat'' = cat' \times cat' =: t$
$cat = \text{corec}\, h\, t$

Catalan numbers

Definition as sequence

$C_0 = 0$
$C_1 = 1$
$C_{n+2} = \sum_{k=1}^{n+1} C_k \cdot C_{n+2-k} = \sum_{k=0}^{n} C_{k+1} \cdot C_{n-k+1}$

Definition as stream

$cat_0 = 0$
$cat_1 = 1 =: h$
$cat'' = cat' \times cat' =: t$
$cat = \text{corec}\, h\, t$

Closed formular

$cat = X + cat \times cat$
$cat = \frac{1-\sqrt{1-4X}}{2}$

Next steps and further work

- Shorter proofs: AC-rewriting and more automation
- Formalize ring based on shuffle product (multiplication wrt. exponential generating functions) using corecursion
- Make simultaneous use of both ring structures (convolution and shuffle product).

Further Reading

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- Herbert S. Wilf.
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  Academic Press, 1990.

Division revised

Characterization

$$(1 + X \times s) \times (1 + X \times s)^{-1} = 1$$
$$(1 + X \times s)_0^{-1} = 1$$
$$(1 + X \times s)_1^{-1} = -s_0 =: h$$
$$((1 + X \times s)^{-1})'' = -\frac{s}{((1 + X \times s)^{-1})'} - s' =: t$$

Corecursive definition

$$((1 + X \times s)^{-1})' = corec\; h\; t\; s$$
$$(1 + X \times s)^{-1} = 1 + X \times ((1 + X \times s)^{-1})'$$
$$v^{-1} = ([v_0^{-1}] \times v)^{-1} \times [v_0^{-1}]$$

Squareroot

Characterization

$$\sqrt{1+X\times s}\times\sqrt{1+X\times s}=1+X\times s$$

$$(\sqrt{1+X\times s})_0=1$$

$$(\sqrt{1+X\times s})_1=\frac{s_0}{2}=:h$$

$$(\sqrt{1+X\times s})''=\frac{s'}{2}-\frac{s\times\sqrt{1+X\times s}'}{4+2X\times\sqrt{1+X\times s}'}=:t$$

Corecursive definition

$$\sqrt{1+X\times s}'=corec\;h\;t\;s$$

$$\sqrt{1+X\times s}=1+X\times\sqrt{1+X\times s}'$$

$$\sqrt{v}=\sqrt{[v_0^{-1}]\times v}\times[\sqrt{v_0}]$$