SAARLAND UNIVERSITY
COMPUTER SCIENCE

Undecidability of the Post Correspondence Problem

Second Bachelor Seminar Talk

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Post Correspondence Problem

$A_{TM} = \{(M, w) \in TM \times \Sigma^* \mid \text{M accepts w }\}$

$$A_{TM} \quad \leq_m \quad PCP$$

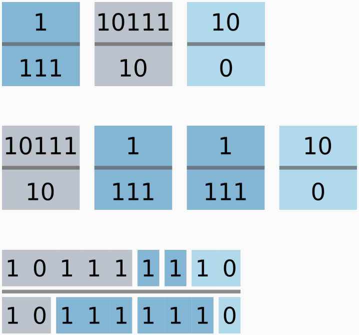
Σ : discrete type

domino := $\Sigma^* \times \Sigma^*$

pcp := finite set of dominos

A list S is a solution of a PCP instance P if

$S \neq \emptyset \wedge S \subseteq P \wedge match\, S$



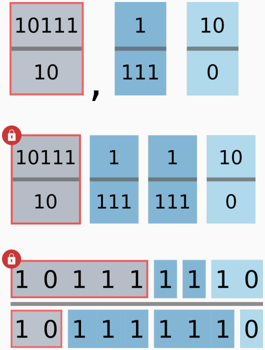
Modified Post Correspondence Problem

$A_{TM} = \{(M, w) \in TM \times \Sigma^* \mid \text{M accepts w } \}$
$PCP = \{P : pcp \mid \exists\, S \,.\, S \neq \emptyset \wedge S \subseteq P \wedge match\, S\}$

$$A_{TM} \quad \leq_m \quad MPCP \quad \leq_m \quad PCP$$

mpcp := domino $\times$ *pcp*

A list S is a solution of an MPCP instance (f, P) if
$S \subseteq P \cup \{F\} \wedge \text{match } (f :: S)$



Modified Post Correspondence ProblemReduction

$$A_{TM} = \{(M, w) \in TM \times \Sigma^* \mid \text{M accepts w } \}$$

$$PCP = \{P : pcp \mid \exists\, S \,.\, S \neq \emptyset \,\wedge\, S \subseteq P \,\wedge\, match\, S\}$$

$$MPCP = \{(f, P) : mpcp \mid \exists\, S \,.\, S \subseteq P \cup \{f\} \,\wedge\, match\,(f :: S)\}$$

$$A_{TM} \quad \leq_m \quad \boxed{MPCP \quad \leq_m \quad PCP}$$

$$f(P) = \left\{ \left[\frac{\star x_1}{y_1 \star}\right], \left[\frac{\star x_2}{y_2 \star}\right], \ldots, \left[\frac{\star x_k}{y_k \star}\right], \left[\frac{\star x_1}{\star y_1 \star}\right], \left[\frac{\#\$}{\$}\right] \right\}$$

Modified Post Correspondence ProblemReduction

$A_{TM} = \{(M, w) \in TM \times \Sigma^* \mid \text{M accepts w } \}$

$PCP = \{P : \mathit{pcp} \mid \exists\, S \,.\, S \neq \emptyset \,\wedge\, S \subseteq P \,\wedge\, \mathit{match}\, S\}$

$MPCP = \{(f, P) : \mathit{mpcp} \mid \exists\, S \,.\, S \subseteq P \cup \{f\} \,\wedge\, \mathit{match}\,(f :: S)\}$

$$\boxed{A_{TM} \qquad \leq_m \quad MPCP} \quad \leq_m \qquad PCP$$

Define a reduction g and prove its correctness

$(M, w) \in A_{TM} \Leftrightarrow g(M, w) \in MPCP$

Formalization of Turing Machines

Turing machine over finite alphabet Σ

$TM := (Q, \delta, q_0, F)$ with

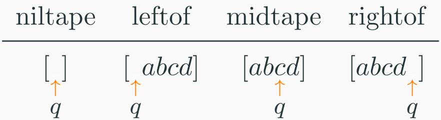
Q : finite type of states

$\delta : Q \times \Sigma_\perp \to$
$Q \times \Sigma_\perp \times \{L, N, R\}$

$q_0 : Q$ initial state

$F \subseteq Q$ set of halting states

```
Inductive tape : Type :=
|niltape : tape
|leftof : Σ → list Σ → tape
|rightof : Σ → list Σ → tape
|midtape : list Σ → Σ → list Σ → tape.
```



niltape	leftof	midtape	rightof
$[\ ]$	$[\ abcd]$	$[abcd]$	$[abcd\ ]$
$\uparrow$ q	$\uparrow$ q	$\uparrow$ q	$\uparrow$ q

A **configuration** consists of the current state and the tape.

TM Acceptability

Inductive predicate $\rightarrow_M^i$ defining reachability

$$\frac{}{c \rightarrow_M^0 c} \qquad\qquad \frac{c_1 \rightarrow_M^i c_2 \qquad \text{state } c_2 \notin F}{c_1 \rightarrow_M^{(Si)} (\text{step } c_2)}$$

TM M accepts configuration c_0 if $\exists\, i\, c_f.\ c_0 \rightarrow_M^i c_f \ \wedge\ \text{state } c_f \in F$

String representation $\langle\cdot\rangle$ of configurations

tape	niltape	leftof	midtape	rightof
c	$[\underset{\underset{q}{\uparrow}}{\ }]$	$[\underset{\underset{q}{\uparrow}}{\ }abcd]$	$[ab\underset{\underset{q}{\uparrow}}{c}d]$	$[abcd\underset{\underset{q}{\uparrow}}{\ }]$
$\langle c\rangle$	$[\mathbf{q}\_]$	$[\mathbf{q}\_abcd]$	$[ab\mathbf{q}cd]$	$[abcd\mathbf{q}]$

Constructing an MPCP Match

Example: Turing machine T accepts all inputs with an even number of a-symbols, replacing them with x.

$$[\underset{\substack{\uparrow\\ q_0}}{a}\,ba] \to [x\,\underset{\substack{\uparrow\\ q_1}}{b}\,a] \to [xb\,\underset{\substack{\uparrow\\ q_1}}{a}\,] \to [xbx\,\underset{\substack{\uparrow\\ q_0}}{}\,] \to [xb\,\underset{\substack{\uparrow\\ q_f}}{x}\,]$$

C_I $\left[\frac{}{\star q_0 aba}\right]$

C_C $\left[\frac{\star}{\star}\right]$ and $\left[\frac{s}{s}\right]$ $\forall s \in \Sigma$

C_T $\left[\frac{q_0 a}{x q_1}\right], \left[\frac{q_1 b}{b q_1}\right], \left[\frac{q_1 a}{x q_0}\right], \left[\frac{x q_0 \star}{q_f x \star}\right], \ldots$

C_D $\left[\frac{s\, q_f}{q_f}\right], \left[\frac{q_f\, s}{q_f}\right]$ $\forall s \in \Sigma$

C_F $\left[\frac{q_f \star}{}\right]$

$$\left[\frac{}{\star q_0 aba}\right]\left[\frac{\star}{\star}\right]\left[\frac{q_0 a}{x q_1}\right]\left[\frac{b}{b}\right]\left[\frac{a}{a}\right]\left[\frac{\star}{\star}\right]\left[\frac{x}{x}\right]\left[\frac{q_1 b}{b q_1}\right]\left[\frac{a}{a}\right]\left[\frac{\star}{\star}\right]\left[\frac{x}{x}\right]\left[\frac{b}{b}\right]\left[\frac{q_1 a}{x q_0}\right]\left[\frac{\star}{\star}\right]\left[\frac{x}{x}\right]\left[\frac{b}{b}\right]\left[\frac{x q_0 \star}{q_f x \star}\right] \to$$

$$\to \left[\frac{\star}{\star}\right]\left[\frac{x}{x}\right]\left[\frac{b}{b}\right]\left[\frac{q_f\, x}{q_f}\right]\left[\frac{\star}{\star}\right]\left[\frac{x}{x}\right]\left[\frac{b q_f}{q_f}\right]\left[\frac{\star}{\star}\right]\left[\frac{x q_f}{q_f}\right]\left[\frac{\star}{\star}\right]\left[\frac{q_f \star}{}\right]$$

$$\frac{\star\, q_0 aba \star x q_1 ba \star x b q_1 a \star x b x q_0 \star x b q_f x \star x b q_f \star x q_f \star q_f \star}{\star\, q_0 aba \star x q_1 ba \star x b q_1 a \star x b x q_0 \star x b q_f x \star x b q_f \star x q_f \star q_f \star}$$

Transforming a TM into an MPCP Instance

TM M accepts configuration $c_0 \Leftrightarrow g(M, c_0) \in MPCP$

Definition of MPCP dominos

C_I	fixed initial card	$\left[\frac{}{\star\langle c_0\rangle}\right]$
C_C	copy cards	$\left[\frac{\star}{\star}\right]$ and $\left[\frac{s}{s}\right]$ $\forall s \in \Sigma$
C_T	transition cards	e.g. $\left[\frac{q_1 a}{x q_2}\right]$ if $q_1 \in Q \setminus F \wedge \delta(q_1, a) = (q_2, x, R)$
C_D	deletion cards	$\left[\frac{s\,q}{q}\right], \left[\frac{q\,s}{q}\right]$ $\forall s \in \Sigma \cup \{\_\}, \forall q \in F$
C_F	final card	$\left[\frac{q\star}{}\right]$ $\forall q \in F$

$g(M, c_0) := (C_I\ , C_C \cup C_T \cup C_D \cup C_F)$

Transition Dominos

$\forall\, q_1 \notin F,\ q_2 \in Q,\ a\ b\ z \in \Sigma$

$\delta(q_1, a) = (q_2, b, L)$ $\left[\dfrac{\star q_1 a}{\star q_2 \_ b}\right]$ and $\left[\dfrac{z q_1 a}{q_2 z b}\right]$

$\delta(q_1, \_) = (q_2, b, R)$ $\left[\dfrac{\star q_1 \_}{\star b q_2}\right]$ and $\left[\dfrac{q_1 \star}{b q_2 \star}\right]$

$\delta(q_1, \_) = (q_2, \_, N)$ $\ldots$

Correctness Proof

$$\forall M\, c_0.\ \exists\, c_f\, i\, .\, c_0 \rightarrow_M^i c_f \wedge (state\, c_f) \in F \ \Leftrightarrow \\ \exists P \subseteq TM_{cards}\, .\, match\left(\left[\frac{}{\star\langle c_0\rangle}\right] :: P\right)$$

Proof direction ⇒ with induction on i:

$i = 0 \wedge (state\, c_0) \in F$

We remove all symbols to the left and to the right of the state using deletion cards.

Example: $\langle c_0 \rangle = q_0\, a\, b$

$$\left[\frac{}{\star q_0 a b}\right] \left[\frac{\star}{\star}\right] \left[\frac{q_0\, a}{q_0}\right] \left[\frac{b}{b}\right] \left[\frac{\star}{\star}\right] \left[\frac{q_0\, b}{q_0}\right] \left[\frac{\star}{\star}\right] \left[\frac{q_0 \star}{}\right]$$

Correctness $\Rightarrow$ cont.

HAVE

IH: $match\left(\left[\frac{}{\star\langle step\ c_0\rangle}\right] :: A\right)$

IH': $\#_1 A = \star\langle step\ c_0\rangle :: \#_2 A$

$$\#_1(step\_cards\ c_0) = \star\,\langle c_0\rangle$$
$$\#_2(step\_cards\ c_0) = \star\,\langle step\ c_0\rangle$$

WANT

$\exists P,\ match\left(\left[\frac{}{\star\langle c_0\rangle}\right] :: P\right)$

$P := (step\_cards\ c_0) \mathbin{+\!\!+} A$

$$\frac{}{\star\langle c_0\rangle}\ \frac{\#_1(step\_cards\ c_0)}{\#_2(step\_cards\ c_0)}\ \frac{\#_1 A}{\#_2 A}$$

$$\frac{}{\star\langle c_0\rangle}\ \frac{\star\,\langle c_0\rangle}{\star\,\langle step\ c_0\rangle}\ \frac{\#_1 A}{\#_2 A}$$

$$\frac{}{\star\langle c_0\rangle}\ \frac{\star\,\langle c_0\rangle}{\star\,\langle step\ c_0\rangle}\ \frac{\star\langle step\ c_0\rangle\ \#_2 A}{\#_2 A}$$

$step\_cards :\ conf \to list\ domino$

Example: $\langle c_0\rangle = q_0\,a\,b\,c$ and $\delta(q_0, a) = (q_1, x, R)$

$$step\_cards\ c_0 = \left[\frac{\star}{\star}\right]\left[\frac{q_0\,a}{x\,q_1}\right]\left[\frac{b}{b}\right]\left[\frac{c}{c}\right]$$

Correctness $\Leftarrow$

$$\forall P\, c_0 . P \subseteq TM_{cards} \land match\left(\left[\frac{}{\star\langle c_0\rangle}\right] :: P\right) \to \exists\, c_f\, i\, .\, c_0 \to_M^i c_f \land (state\, c_f) \in F$$

Proof:

1. Size induction on $|P|$
2. Case analysis on $state\, c_0 \in F$

$$\begin{aligned}
state\, c_0 \in F :&\ \ c_f := c_0 \\
state\, c_0 \notin F :&\ \ \text{prove that } P \text{ can be split into } (step\_cards\, c_0) +\!\!+ P' \\
&\ \ \text{use IH with } P' \text{ and } step\, c_0 \text{ to get } i \text{ and } c_f \text{ with} \\
&\ \ step\, c_0 \to_M^i c_f \land (state\, c_f) \in F \\
&\ \ c_0 \to_M^1 step\, c_0 \to_M^i c_f \\
&\ \ \mathbf{c_0 \to_M^{(S\, i)} c_f}
\end{aligned}$$

Structure of the Solution List

Assumptions: $P \subseteq TM_{cards}$, $state\ c_0 \notin F$, $match\left(\left[\frac{\quad}{\star\langle c_0\rangle}\right] +\!\!+\, P\right)$

Goal: $P = (step\_cards\ c_0) :: P'$

Example: $c_0 = [\underset{\substack{\uparrow \\ q_0}}{a}\, ba]$ and $step\ c_0 = [x\, \underset{\substack{\uparrow \\ q_1}}{b}\, a]$ with $\delta(q_0, a) = (q_1, x, R)$

$$\left[\frac{\quad}{\star q_0 aba}\right]\left[\frac{\star}{\star}\right]\left[\frac{q_0 a}{x q_1}\right]\left[\frac{b}{b}\right]\left[\frac{a}{a}\right] :: P'$$

$$\frac{\star q_0 aba}{\star q_0 aba \star x q_1 ba}$$

$$P = \left[\frac{\star}{\star}\right]\left[\frac{q_0 a}{x q_1}\right]\left[\frac{b}{b}\right]\left[\frac{a}{a}\right] :: P'$$

$$P = (step\_cards\ c_0) +\!\!+ P'$$

C_I $\left[\frac{\$}{\$ \star \langle c_{start}\rangle}\right]$

C_C $\left[\frac{\star}{\star}\right]$ and $\left[\frac{s}{s}\right]$ $\forall s \in \Sigma$

C_T $\left[\frac{q_0 a}{x q_1}\right]$

C_D $\left[\frac{s\, q}{q}\right], \left[\frac{q\, s}{q}\right]$ $\forall s \in \Sigma \cup \{\_\}, \forall q \in F$

C_F $\left[\frac{q\star}{\quad}\right]$ $\forall q \in F$

Conclusion

What we have

- Formal definitions of single-tape Turing machines, MPCP and PCP
- Verified reductions from A_{TM} to $MPCP$ and $MPCP$ to PCP

Future work

- Undecidability of $\mathcal{L}(G_1) \cap \mathcal{L}(G_2) = \emptyset$ for CFG's G_1, G_2
- Reduction of A_{TM} to the word problem for string rewriting systems (SRS)
- Reduction of SRS to PCP

$$A_{TM} \leq SRS \leq PCP^{1}$$

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Coq Development

	Spec	Proof	Σ
MPCP $\leq$ PCP	130	180	310
$A_{TM} \leq$ MPCP	370	540	910
TM	170	110	280
Σ	670	830	1500

Formalization of TM Configurations

```
Variable Σ : finType.
Variable Q: finType.

Inductive tape : Type :=
  | niltape : tape
  | leftof : Σ → list Σ → tape
  | rightof : Σ → list Σ → tape
  | midtape : list Σ → Σ → list Σ → tape.

Definition tape' : Type := Σ* × Q × Σ*
```

Additional blank symbol needed to express $[\underset{\overset{\uparrow}{q}}{\ }abcd]$ vs. $(q, \mathit{leftof}\ a\, [b; c; d])$