Representations of Boolean Functions in Constructive Type Theory

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Motivation and Goal

- Goal:
  Define a *informative canonical* representation of boolean functions in Coq $\Longrightarrow$ *prime trees*.
- Why:
  Coq's type theory is *intentional*.
- How:
  By showing that boolean functions and prime trees are isomorphic.

$$\textit{Prime Trees} \;\cong\; \textit{Boolean Functions}$$

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- Additional constraint: Mappings must preserve equality.

$$\forall a_1, a_2 \in A : \ a_1 \equiv_A a_2 \Rightarrow g(a_1) \equiv_B g(a_2)$$

$$\forall b_1, b_2 \in B : \ b_1 \equiv_B b_2 \Rightarrow f(b_1) \equiv_A f(b_2)$$

The notion of isomorphism in Coq

- Use *setoids* and *morphisms* for elegant definition:

$$\begin{aligned} Setoid \quad &:= \quad \{(T : Type, \ \equiv_T : T \times T) \mid \ \equiv_T \ \ ER\} \\ (A, \equiv_A) \twoheadrightarrow (B, \equiv_B) \quad &:= \quad \{f : A \to B \mid f \text{ preserves } \equiv_B\} \end{aligned}$$

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- $f : (A, \equiv_A) \twoheadrightarrow (B, \equiv_B)$ and $g : (B, \equiv_B) \twoheadrightarrow (A, \equiv_A)$ form a *setoid-isomorphism* iff

$$\begin{aligned} \forall b : B, &\quad (f \circ g)(b) \equiv_B b \\ \forall a : A, &\quad (g \circ f)(a) \equiv_A a \end{aligned}$$

Boolean functions in theory

- Finite set of variables: V
- Assignments (σ): $V \rightarrow bool$
- Boolean functions (ϕ, ψ): $(V \rightarrow bool) \rightarrow bool$

Decision trees in theory

- Based on *conditionals*:

$$(x, s, t) \;:=\; (x \wedge s) \vee (\neg x \wedge t)$$

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  - $\top$ and $\perp$ are decision trees.
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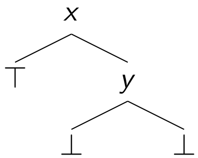
  - $\top$ and $\bot$ are decision trees.
  - (x, s, t) is a decision tree iff x variable and s, t decision trees.

- Tree interpretation:

$$(x, \top, (y, \bot, \bot)) \implies$$



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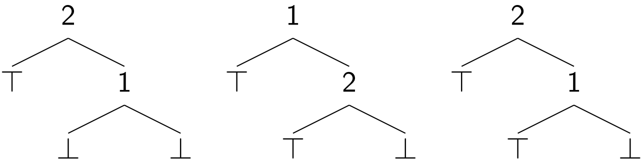
- Examples: Let $V := \{1, 2\}$



ordered not reduced	not ordered reduced	ordered reduced $\Rightarrow$ prime tree

Roadmap

- Whatever definitions used,
  - $\mathcal{BF} :=$ Boolean functions
  - $\mathcal{DT} :=$ Decision trees
  - $\mathcal{PT} :=$ Prime trees

  we will need:
  - Decidable equality: $\forall t_1 t_2 : \mathcal{DT},\ \{t_1 = t_2\} + \{t_1 \neq t_2\}$
  - Denotational completeness: $\forall \phi : \mathcal{BF},\ \{t : \mathcal{PT} \mid [\![t]\!] \equiv \phi\}$
  - Core result : $\forall t_1 t_2 : \mathcal{PT},\ t_1 \neq t_2 \to [\![t_1]\!] \not\equiv [\![t_2]\!]$
- Morphisms:
  - Denotational Completeness (ex. V4) : $(\mathcal{BF}, \equiv) \twoheadrightarrow (\mathcal{PT}, =)$
  - Denotational Semantics $[\![\cdot]\!]$: $(\mathcal{PT}, =) \twoheadrightarrow (\mathcal{BF}, \equiv)$
- Isomorphism:

$$(\mathcal{BF}, \equiv) \cong (\mathcal{PT}, =)$$

- Cascaded boolean functions:

$$bool^n \rightarrow bool$$

Version 1: The dependently typed approach

Boolean functions in Coq

- Cascaded boolean functions:

$$bool^n \rightarrow bool$$

- `Fixpoint` $nfun\ A\ n\ B$:=

  `match` n `with`

  $\mid 0 \Rightarrow B$

  $\mid S\ n \Rightarrow A \rightarrow (nfun\ A\ n\ B)$

  `end.`

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```
match n with
  | 0 => B
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end.
```

- Equivalence:

$$\phi \equiv \psi \;:=\; \forall \vec{x},\ \phi\ \vec{x} = \psi\ \vec{x}$$

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Decision trees in Coq

- `Inductive` DT : $nat \rightarrow$ `Type` :=
  $\mid DT_0 : DT\ 0$
  $\mid DT_1 : DT\ 0$
  $\mid DT_I : \forall \{n\}, DT\ n \rightarrow DT\ n \rightarrow DT\ (S\ n)$
  $\mid DT_L : \forall \{n\}, DT\ n \rightarrow DT\ (S\ n).$
- Dependency indicates number of variables the tree depends on.

- Denotational Semantics: Via recursion on the decision tree.

$$
\begin{array}{rcl}
[\![\bot]\!] & = & \mathit{false} \\
[\![\top]\!] & = & \mathit{true} \\
[\![(\_, t_1, t_2)]\!] & = & \lambda b : \mathit{bool}. \begin{cases} [\![t_1]\!], & b = \mathit{true} \\ [\![t_2]\!], & b = \mathit{false} \end{cases} \\
[\![t^1]\!] & = & \lambda \_ : \mathit{bool}.\ [\![t]\!]
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Denotational semantics and prime trees

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- Members of $DT\ \ n$ are already ordered by design.
- Thus prime trees are defined by

$$PT\ \ n \ := \ \{t : DT\ \ n \mid \mathit{reduced}\ \ t\}$$

- We need to write an *inversion* function ourselves:

  Definition $DT\_Inv\ \{n : nat\}\ (t : DT\ n)$:
   match n as z return $DT\ z \rightarrow$ Type with
    $|\ O \Rightarrow$ fun $t \Rightarrow \{t = \bot\} + \{t = \top\}$
    $|S\ n' \Rightarrow$ fun $t \Rightarrow$

  $$\{p : (DT\ n') \times (DT\ n') \mid t = (\_, fst\ p, snd\ p)\} + \{dt : DT\ n' \mid t = dt^1\}$$

   end t.

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Decidable equality

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  $\qquad\qquad\qquad + \{dt : DT\ n' \mid t = dt^1\}$
  $\quad \texttt{end}\ t.$

- Decidable equality

  $$\forall t_1 t_2 : DT\ n,\ \{t_1 = t_2\} + \{t_1 \neq t_2\}$$

  by recursion on n.

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- Solution: Recursive definition instead of inductive definition
- Fixpoint DT_{rec} $(n : nat)$: Type :=

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  end.
- No complicated inversion function needed.

Version 1 and 2: Rest of roadmap

- Denotational Completeness

$$\forall \phi : bool^n \to bool,\ \{t : PT\ n \mid [\![t]\!] \equiv \phi\}$$

by recursion on number of arguments n.

Version 1 and 2: Rest of roadmap

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- Both straightforward.

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- Copies!
  - Boolean functions: *true*, $\lambda\_.true$, $\lambda\_\_.true$, ...
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- Remedy: Recursive Decision trees.
  - `injection` works perfectly $\Rightarrow$ shorter proofs.
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- Remedy: Recursive Decision trees.
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- Actually pretty convenient to work with.

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- Decidable equality comes for free: `decide equality`.

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Sematics and prime trees

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- Unfortunately not ordered by design:

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SPT \ := \ \{t : SDT \mid \mathit{reduced}\ t \wedge \mathit{ordered}\ t\}
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- Only one constant true function : $\lambda\_ : (\mathit{nat} \to \mathit{bool}).\ \mathit{true}$
- Infinitely many variables $\Rightarrow$ infinite decision trees!
- Restriction to only the *continuous* boolean functions

Continuous boolean functions

- $cts_n\ \phi\ :=$ "it suffices to consider the first n variables to evaluate ϕ".
  - n is a *modulus of continuity*.
- $cts\ \phi\ := \exists n : nat,\ cts_n\ \phi$

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  impossible to obtain $\Longrightarrow$ *Elim restriction*.
- 3 possibilities to circumvent the elim restriction.

Version 3: Using the Axiom of Continuity
Denotational Completeness

- Drastic solution:
  - $ctsT\ \phi\ := \ \{n : nat \mid cts_n\ \phi\}$
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- Denotational completeness by recursion on the modulus of continuity given by CTS.
- CTS inconsistent with $CDP := \forall P : Prop,\ \{P\} + \{\neg P\}$

$$CTS \rightarrow CDP \rightarrow False$$

Version 3: Using the Axiom of Continuity
Core Result

- Direct proof of

$$\forall t_1 t_2 : SPT,\ t_1 \neq t_2 \rightarrow [\![t_1]\!] \neq [\![t_2]\!]$$

via `induction` is **HUGE** : 9 cases!

- Direct proof of

$$\forall t_1 t_2 : SPT,\ t_1 \neq t_2 \rightarrow [\![t_1]\!] \not\equiv [\![t_2]\!]$$

  via `induction` is **HUGE** : 9 cases!

- Use *size induction* on pairs of decision trees.
  - Divide proof into *case analysis*
    - $|t_1| + |t_2| = 0 \rightarrow (t_1 = \top \vee t_1 = \bot) \wedge (t_2 = \top \vee t_2 = \bot)$.
    - $|t_1| + |t_2| = m + 1 \rightarrow \begin{cases} t_1 = (n, t, t') & \wedge\ n \notin t_2 \\ n \notin t_1 & \wedge\ t_2 = (n, t, t') \\ t_1 = (n, t_1', t_1'') & \wedge\ t_2 = (n, t_2', t_2'') \end{cases}$

  and *main proof*:
    - $|t_1| + |t_2| = 0$ case is trivial.
    - $|t_1| + |t_2| = m + 1$ cases (3) of moderate difficulty.

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- Prove denotational completeness as proposition

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$$\forall \phi : BF_{cts},\ \exists t : SPT,\ [\![t]\!] \equiv \phi$$

- Prove core result like before

$$\forall t_1 t_2 : SPT,\ t_1 \neq t_2 \to [\![t_1]\!] \not\equiv [\![t_2]\!]$$

- From denotational completeness and core result derive that there is a unique SPT for every BF_{cts}:

$$\forall \phi : BF_{cts},\ \exists! t : SPT,\ [\![t]\!] \equiv \phi$$

Version 4: Using the Axiom of Description

A morphism from $(BF_{cts}, \equiv)$ to (SPT,=)

- From denotational completeness and core result derive that there is a unique *SPT* for every BF_{cts}:

$$\forall \phi : BF_{cts},\ \exists! t : SPT,\ [\![t]\!] \equiv \phi$$

- Turn this proof into a mapping using *Axiom of Description*:

$$\forall (T : Type)(P : T \to Prop),\ (\exists! t : T, P\ t) \to \{t : T \mid P\ t\}$$

- We want to use $[\![\cdot]\!]$

Version 4: Using the Axiom of Description

A morphism from (SPT,=) to (BF_{cts}, $\equiv$)

- We want to use $[\![\cdot]\!]$
- Prove that decision trees describe continuous functions

$$\forall t : SDT,\ \ cts\ [\![t]\!]$$

by writing a function that determines a modulus of continuity

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$$\forall t : SDT,\ \ ctsT\ [\![t]\!]$$

- Modulus of continuity is largest variable in the tree.

- Pair boolean functions with their modulus of continuity.

$$BF_{ctsT} \;:=\; \{\phi : BF \;\&\; cts_T\ \phi\}$$

- Equivalence:

$$(\phi, n) \equiv (\psi, m) \;:=\; \phi \equiv \psi$$

Version 5: The No-Axiom Version

Boolean functions as dependent pairs

- Pair boolean functions with their modulus of continuity.

$$BF_{ctsT} \;:=\; \{\phi : BF \;\&\; cts_T\;\phi\}$$

- Equivalence:

$$(\phi, n) \equiv (\psi, m) \;:=\; \phi \equiv \psi$$

- Denotational Completeness

$$\forall(\phi, n) : BF_{ctsT},\; \{t : SPT \mid [\![t]\!] \equiv \phi\}$$

  by recursion on modulus of continuity n.
- Core result as before.

Version 3,4 and 5: Summary

- No copies!
- More work:
  - Ordering
  - $n \in t$
  - Lemmas relating orderedness and variable occurrences.
- Axioms (Versions 3 and 4)!

Things to improve

- Goal: Canonical representation for boolean functions

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- Setoid-isomorphism *not* meaning preserving!

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- Definition of *representative* of boolean functions:

$$\left\{ \begin{array}{lcl} T & : & \mathit{Type} \\ =_T & : & T \to T \to \mathit{Prop} \\ [\![\cdot]\!] & : & T \to \mathcal{BF} \\ ER & : & =_T \text{ is equivalence relation} \\ P & : & \forall t_1\, t_2 : T,\; t_1 =_T t_2 \to [\![t_1]\!] \equiv_{\mathcal{BF}} [\![t_2]\!] \end{array} \right\}$$

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- *Meaning preserving* morphisms:

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- Used setoids are representatives, morphisms are meaning preserving

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