Clausal Graph Tableaux for Hybrid Logic with Eventualities and Difference

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Abstract

- Framework for tableau-style decision procedures
- Modal logic with star modalities, difference modalities, and nominals
- Novel tableau system called clausal graph tableaux, in the spirit of Pratt's graph tableaux [1980], different from the usual tree tableaux
- First time that graph tableaux are adapted to a logic with nominals (or difference modalties)

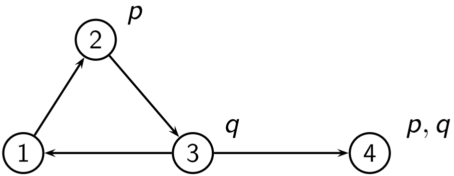
Related work, star modalities

- Star modalities ($\Diamond^*$, $\Box^*$) express properties of nodes reachable from the current node
- Star modalities are a prominent feature of PDL [Fischer/Ladner 1979] and temporal logics
- Star modalities yield a non-compact logic
- First tableau-style decision procedure for PDL devised by Pratt [1980], graph tableaux, worst-case optimal
- Goré and Widmann [IJCAR 2010] develop efficient prover for PDL with converse, algorithmic refinement of Pratt's approach

Related work, nominals and difference modalities

- Nominals are predicates that hold for exactly one node
- Nominals are the distinguishing feature of hybrid logic
- Nominals are also a prominent feature of description logics
- Difference modalities (D and $\bar{\mathsf{D}}$) express properties of nodes different from the current node [de Rijke 1992]
- Difference modalities can express global modalities and nominals
- Terminating tableau systems for hybrid logic with golbal modalities devised by Bolander, Braüner, and Blackburn [2006,2007]
- First terminating tableau system for difference modalities devised by Kaminski and Smolka [2008], prefixed tree tableaux
- First terminating tableau system for eventualities and nominals [KS, IJCAR 2010], clausal tree tableaux

Models



- Directed Graphs (nodes, edges)
- Nodes are labelled with predicates (p, q, ...)

Formulas

$$s \;::=\; p \mid \neg s \mid s \wedge s \mid \Diamond s \mid \Diamond^* s \mid \mathrm{D}s \mid x$$
$$\mid s \vee s \mid \Box s \mid \Box^* s \mid \bar{\mathrm{D}}s$$

- $M, a \models p$ node a is labelled with p
- $M, a \models \Diamond s$ some successor of a satisfies s
- $M, a \models \Diamond^* s$ some node reachable from a satisfies s
- $M, a \models \mathrm{D}s$ some node different from a satisfies s

- Nominals x: predicates satisfied by exactly one node

Expressivity of difference modality

- There exists a node that satisfies s: $\mathsf{E}s \equiv s \vee \mathsf{D}s$
- Every node satisfies s: $\mathsf{A}s \equiv s \wedge \bar{\mathsf{D}}s$
- Exactly one node satisfies s: $\mathsf{N}s \equiv \mathsf{E}(s \wedge \bar{\mathsf{D}}\neg s)$

Normal formulas, literals, and clauses

$$s \;::=\; L \mid s \wedge s \mid s \vee s \qquad\qquad \textit{normal formulas}$$

$$L \;::=\; p \mid \neg p \mid \Diamond s \mid \Box s \mid \Diamond\Diamond^* s \mid \Box\Box^* s \mid \mathsf{D}s \mid \bar{\mathsf{D}}s \qquad \textit{literals}$$

- Normal formulas are negation normal and star normal
- NF of formula can be obtained in linear time (graph representation)
- $\Diamond^* s \equiv s \vee \Diamond\Diamond^* s$ and $\Box^* s \equiv s \wedge \Box\Box^* s$
- DNF of normal formula does not introduce new literals
- Clause: Finite set of literals not containing complementary pairs
- Clauses are interpreted conjunctively

Demos

- Demos are syntactic models, like Herbrand models for FOL
- A demo is a finite and nonempty set of clauses satisfying certain decidable properties
- A demo describes a model whose nodes are the clauses of the demo
- The model M described by a demo Δ satisfies all clauses of Δ
- More precisely: $M, C \models C$ for all $C \in \Delta$

Bounded model theorem

- Theorem: For every satisfiable formula there exists a demo Δ such that the model described by Δ satisfies the formula and Δ employs only literals that occur in the NF of the formula
- Proof
- Let M be a model of a formula s
- $C \in \Delta$ iff M has a node a such that C consists of all literals in the NF of s that hold at a
- Note: The nodes of M map to the clauses of Δ
- Model described by Δ satisfies s (follows by Demo Lemma shown later)
- Existential difference literals require auxiliary nominals (shown later)

Tableaux and goal-directed search of demos

- Goal-directed search of demos is possible
- Start with clauses describing a DNF of input formula
- Add clauses according to tableau rules
- Leads to demo of input formula if input formula is satisfiable
- Yields decision procedure since closure obtained with tableau rules is finite
- Note: A tableau is just a set of clauses, no branches, no links

Example 1

1. $\Diamond\Diamond^* p \;\wedge\; \Box\neg p \;\wedge\; \Box\Box\neg p$ $\Diamond\Diamond^* p,\; \Box\neg p,\; \Box\Box\neg p$
2. $(p \vee \Diamond\Diamond^* p) \;\wedge\; \neg p \;\wedge\; \Box\neg p$ $\Diamond\Diamond^* p,\; \neg p,\; \Box\neg p$
3. $(p \vee \Diamond\Diamond^* p) \;\wedge\; \neg p$ $\Diamond\Diamond^* p,\; \neg p$
4. $p \vee \Diamond\Diamond^* p$ p
5. $\Diamond\Diamond^* p$

Clauses 1, 2, 3, 4 comprise a demo that yields a model as follows:

$$1 \to 2 \to 3 \to 4_p$$

Example 2

1. $\Diamond\Diamond^* p,\ \neg p,\ \Box(x \wedge \neg p),\ \Diamond\Box\neg p$
2. $\Diamond\Diamond^* p,\ x,\ \neg p$
3. p
4. $\Diamond\Diamond^* p$
5. $\Box\neg p,\ x,\ \neg p$
6. $\Diamond\Diamond^* p,\ x,\ \neg p,\ \Box\neg p$ *obtained by taking union of clauses 2, 4*
7. $\Diamond\Diamond^* p,\ \neg p$

Clauses 1, 3, 5, 6 comprise a demo that yields a model as follows:

$$1 \longrightarrow 5_x \longrightarrow 6 \longrightarrow 3_p$$

Example 3

1. $\bar{\mathrm{D}}\mathrm{D}p,\ \Diamond p,\ \neg p$
2. p
3. $p,\ \mathrm{D}p$
4. $p,\ x$ *x is auxiliary nominal for* $\mathrm{D}p$
5. $p,\ x,\ \mathrm{D}p$

Clauses 1, 3, 5 comprise a demo that yields a model as follows:

$$1 \longrightarrow 3_p \qquad 5_{p,x}$$

Support, request, and links

- C supports s: C contains a clause of the DNF of s
- Support implies logical entailment
- Sharpened BMT: one clause of demo supports formula
- Request of C: conjunction of all formulas t such that $\Box t \in C$
- Link: Triple CsC' such that $\Diamond s \in C$
  and C' supports s and request of C
- A link CsC' describes an edge (C, C') as required by $\Diamond s \in C$

Definition of demos

A set Δ of clauses is a demo if it satisfies the following conditions:

- If $\Diamond s \in C \in \Delta$, then $\exists C' \in \Delta$ such that C' supports s and $\mathcal{R}C$
- If x is a nominal, then there is exactly one $C \in \Delta$ such that $x \in C$
- If $\mathsf{D}s \in C \in \Delta$, then $\exists C' \in \Delta$ such that $C' \neq C$ and C' supports s
- If $\bar{\mathsf{D}}s \in C \in \Delta$ and $C' \in \Delta$ such that $C \neq C'$, then C' supports s
- If $\Diamond\Diamond^* s \in C \in \Delta$, then $\exists C_1, \ldots, C_n \in \Delta$ such that:
  - $C_1 = C$, $n \geq 2$, C_n supports s
  - $\forall i \in [1, n-1] :\ C_i(\Diamond^* s)C_{i+1}$ is a link

Demo Lemma

Let Δ be a demo and M be the following model:

- The nodes of M are the clauses of Δ
- p labels C iff $p \in C$
- (C, C') is an edge of M iff CsC' is a link for some s

Then $M, C \models L$ for every $C \in \Delta$ and every $L \in C$.

Proof idea. Show by induction on formulas s:
$\forall C \in \Delta$: if C supports s, then $M, C \models s$

Tableau closure

- Set of clauses obtained from DNF of input formula with the rules

$$\frac{\Diamond s \in C \in T}{\mathcal{D}(s \wedge \mathcal{R}C) \subseteq T}$$

$$\frac{x \text{ nominal}}{\{x\} \in T} \qquad \frac{x \in C \in T \qquad x \in C' \in T}{\mathcal{D}(C \wedge C') \subseteq T}$$

$$\frac{\mathsf{D}s \in C \in T}{\mathcal{D}(s \wedge (x \vee \neg x)) \subseteq T} \; x \text{ auxiliary nominal for } Ds$$

$$\frac{\bar{\mathsf{D}}s \in C \in T \qquad C' \in T}{\mathcal{D}(C \wedge C') \cup \mathcal{D}(C' \wedge s) \subseteq T} \; C \neq C' \text{ and } C' \text{ doesn't support } s$$

- Finite since no new literals are added (literals from NF of input formula plus auxiliary literals for existential difference literals)

Completeness theorem

- Theorem: The tableau closure of a satisfiable formula contains a demo of the formula
- Proof
- Let M be a model of a formula s and T be the tableau closure of s
- Show: $\exists$ demo $\Delta \subseteq T$ $\exists$ clause $C \in \Delta$ such that C supports s
- A clause $C \in T$ is prominent if there exists a state a in M that satisfies C and all other clauses in T that satisfy a are contained in C
- The set of all prominent clauses is a demo
- Since M satisfies one of the initial clauses, demo contains a clause that contains an initial clause
- Claim follows with Demo Lemma

Practical decision procedures

- So far we have a framework for decision procedures
- Algorithmic refinements are needed
- Incremental computation of DNF
- How does procedure know that a demo for one of the inital clauses is found?
- Nice solution for the case without nominals and difference [Goré and Widmann, IJCAR 2010] (PDL with converse)
  - Rules that determine unsatisfiable clauses (pruning of search space)
  - Rules that determine satisfication/dissatisfaction of eventualities ($\Diamond\Diamond^* s$)
- Tree tableau solution for general case [Kaminski and Smolka, IJCAR 2010] (no difference modalities)
- Open problem: Practical, worst-case optimal procedure for the case with nominals (not possible with tree tableaux)

Tree tableaux versus graph tableaux

- Tree tableaux
  - branch on disjuntive formulas
  - have property that all clauses of a succesful branch are satisfied by a single model
  - adapt easily to nominals
  - are not worst-case optimal
  - need only one successor for diamond literals
  - simplify eventuality checking, bad loop optimization
  - are used by the provers Fact and Spartacus
- Graph tableaux
  - do not branch, thus avoid recomputation
  - are typically worst-case optimal
  - were pioneered by Pratt 1980
  - are implemented with complex control structures
    [Goré and Widmann, IJCAR 2010]

Incremental DNF computation

- Consider $(p_1 \vee q_1) \wedge \ldots \wedge (p_n \vee q_n) \wedge \Diamond s$
- DNF has 2^n clauses
- DNF not needed if s unsatisfiable
- Compute DNF lazily, depth first search, backjumping
- Auxiliary clauses containing disjunctions

Weak and strong DNF

- Consider $p \wedge (q_1 \vee q_2)$
- Weak DNF: $\{p, q_1\}$, $\{p, q_2\}$
- Strong DNF: $\{p, q_1\}$, $\{p, \neg q_1, q_2\}$ (semantic branching)
- Strong DNF for demo search
- Weak DNF for testing support

Summary

- Modal logic with nominals and star and difference modalities
- Syntactic class of models called demos
- Bounded model theorem: Every satisfiable formula is satisfied by a demo obtained from the literals of the NF of the formula
- Tableau closure providing for goal-directed demo search
- Bridge from filtration to tableaux
- Open issues concerning practical decision procedures