An Almost Constructive Proof of Classical First-Order Completeness

First Bachelor Seminar Talk

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Partial History of First-Order Completeness

1928 — First formal statement by Hilbert and Ackermann[1]

1929 — First proven by Gödel[2]

1947 — Greatly simplified by Henkin[3]

⋮

2016 — Constructive analysis by Herbelin and Ilik [4]

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[1] Ackermann and Hilbert. "Grundzüge der theoretischen Logik"

[2] Gödel. "Über die Vollständigkeit des Logikkalküls"

[3] Henkin. "The Completeness of the First-Order Functional Calculus"

[4] Herbelin and Ilik. *An analysis of the constructive content of Henkin's proof of Gödel's completeness theorem*

Definition (Syntax)

$$s, t : \mathtt{T} \;::=\; e \mid f\ t \mid x \mid p \qquad\qquad x, p : \mathbf{N}$$

$$\varphi, \psi : \mathtt{F} \;::=\; \dot{\bot} \mid P\ s\ t \mid \varphi \mathbin{\dot{\to}} \psi \mid \dot{\forall} x.\varphi \qquad\qquad x : \mathbf{N}$$

$$\dot{\neg}\varphi := \varphi \mathbin{\dot{\to}} \dot{\bot} \qquad \dot{\exists} x.\varphi := \dot{\neg}\dot{\forall} x.\dot{\neg}\varphi \qquad \varphi \mathbin{\dot{\vee}} \psi := \dot{\neg}\varphi \mathbin{\dot{\to}} \psi$$

Definition (Deduction system)

$$\textsc{Ctx}\,\frac{\varphi \in A}{A \vdash \varphi} \qquad \text{II}\,\frac{\varphi :: A \vdash \psi}{A \vdash \varphi \mathrel{\dot{\rightarrow}} \psi}$$

$$\text{IE}\,\frac{A \vdash \varphi \mathrel{\dot{\rightarrow}} \psi \qquad A \vdash \varphi}{A \vdash \psi} \qquad \text{DN}\,\frac{A \vdash \dot{\neg}\dot{\neg}\varphi}{A \vdash \varphi}$$

$$\textsc{AllI}\,\frac{A \vdash \varphi^x_p \quad p \text{ fresh for } \varphi \text{ and } A}{A \vdash \dot{\forall} x.\varphi}$$

$$\textsc{AllE}\,\frac{A \vdash \dot{\forall} x.\varphi \quad t \text{ closed}}{A \vdash \varphi^x_t}$$

Definition (Interpretation)

An interpretation **I** on a domain **D** consists of:

$$e^{\mathbf{I}} : \mathbf{D} \qquad f^{\mathbf{I}} : \mathbf{D} \to \mathbf{D} \qquad \cdot^{\mathbf{I}} : \mathbf{N} \to \mathbf{D} \qquad P^{\mathbf{I}} : \mathbf{D} \to \mathbf{D} \to \mathbf{P}$$

Definition (Evaluation)

Given $\rho : \mathbf{N} \to \mathbf{D}$, we extend **I** to $t^{\mathbf{I},\rho} : \mathbf{D}$ and $\rho \vDash_{\mathbf{I}} \varphi : \mathbf{P}$:

$$\rho \vDash_{\mathbf{I}} \dot{\bot} = \bot$$
$$\rho \vDash_{\mathbf{I}} P\ s\ t = P^{\mathbf{I}}\ s^{\mathbf{I},\rho}\ t^{\mathbf{I},\rho}$$
$$\rho \vDash_{\mathbf{I}} \varphi \dot{\to} \psi = \rho \vDash_{\mathbf{I}} \varphi \to \rho \vDash_{\mathbf{I}} \psi$$
$$\rho \vDash_{\mathbf{I}} \dot{\forall} x.\varphi = \forall d : \mathbf{D}.\ \rho[x \mapsto d] \vDash_{\mathbf{I}} \varphi$$

$$A \vDash \varphi \ := \ \forall \mathbf{I}\ \rho.\ \rho \vDash_{\mathbf{I}} A \to \rho \vDash_{\mathbf{I}} \varphi$$

Definition (Theories)

We extend the previous notions to theories $\mathcal{T} : \mathtt{F} \to \mathbf{P}$:

$$\mathcal{T} \vDash \varphi := \forall \mathbf{I}\, \rho.\ \rho \vDash_{\mathbf{I}} \mathcal{T} \to \rho \vDash_{\mathbf{I}} \varphi$$
$$\mathcal{T} \vdash \varphi := A \vdash \varphi \exists A.\ A \subseteq \mathcal{T} \land A \vdash \varphi$$

Definition (Consistency)

We call $\mathcal{T} : \mathtt{F} \to \mathbf{P}$

- consistent if $\mathcal{T} \nvdash \dot{\bot}$
- maximally consistent if $\mathcal{T} \nvdash \dot{\bot}$ and $\varphi \in \mathcal{T}$ if $\mathcal{T} \cup \{\varphi\} \nvdash \dot{\bot}$

Proof Outline

$$\boxed{\underset{\text{consistent}}{\mathcal{T}} \xrightarrow{\text{Model Existence}} \; ? \vDash_? \mathcal{T}}$$

$$\Downarrow$$

$$A \vDash \varphi \to A \vdash \varphi$$

Quantifier-free Model Existence

$$\begin{tikzcd}[column sep=huge] \mathcal{T} \arrow[r, "\text{Lindenbaum}"] & \Omega \arrow[r, "\text{Herbrandt}"] & \vDash_\Omega \end{tikzcd}$$

consistent closed	maximally consistent	model for $\mathcal{T}$

Definition

Given a consistent $\mathcal{T}$, we fix an enumeration $\mathcal{E}_{\mathtt{F}}$ and define

$$\Omega_0 = \mathcal{T} \qquad \Omega_{n+1} = \begin{cases} \Omega_n \cup \{\mathcal{E}_{\mathtt{F}}\ n\} & \Omega_n \cup \{\mathcal{E}_{\mathtt{F}}\ n\} \text{ consistent} \\ \Omega_n & \text{otherwise} \end{cases}$$

$$\Omega := \bigcup \Omega_n$$

Lemma (Lindenbaum)

Ω is a maximally consistent extension of $\mathcal{T}$.

Quantifier-free Model Existence

$$
\begin{tikzcd}
\mathcal{T} \arrow[r, "\text{Lindenbaum}"] & \Omega \arrow[r, "\text{Herbrandt}"] & \vDash_\Omega
\end{tikzcd}
$$

consistent closed	maximally consistent	model for $\mathcal{T}$

Definition (Herbrandt model)

Given a theory Ω we define its Herbrandt model on closed terms $\mathtt{T}^c$:

$$t^{\Omega,\rho} := t \qquad\qquad P^{\Omega}\ s\ t := P\ s\ t \in \Omega$$

Lemma (Model correctness)

Let Ω be maximally consistent and φ be closed and quantifier-free, then

$$\vDash_{\Omega} \varphi \ \leftrightarrow\ \varphi \in \Omega$$

Corollary (Model existence)

Let $\mathcal{T}$ be consistent and closed, then $\vDash_{\Omega} \mathcal{T}$.

Lemma (Maximally consistent membership)

Let Ω be maximally consistent. Then $\varphi \in \Omega \;\leftrightarrow\; \Omega \vdash \varphi$.

Lemma (Model correctness)

Let Ω be maximally consistent and φ be closed and quantifier-free, then

$$\vDash_\Omega \varphi \;\leftrightarrow\; \varphi \in \Omega$$

Proof.

Proof per induction on the size of φ. There are three cases:

- $P\,s\,t \in \Omega \;\leftrightarrow\; P\,s\,t \in \Omega$
- $\bot \;\leftrightarrow\; \Omega \vdash \dot{\bot}$
- $(\Omega \vdash \varphi \rightarrow \Omega \vdash \psi) \;\leftrightarrow\; \Omega \vdash \varphi \mathbin{\dot{\rightarrow}} \psi$

$\square$

First-Order Model Existence

$$\begin{array}{ccccccc} & \text{Henkin} & & \text{Lindenbaum} \\ & & & \text{Herbrandt} & \\ \mathcal{T} & \xrightarrow{\hspace{8em}} & \mathcal{H} & \xrightarrow{\hspace{4em}} & \Omega & \xrightarrow{\hspace{4em}} & \vDash_\Omega \\ \text{consistent} & & \text{consistent} & & & & \text{model for } \mathcal{T} \\ \text{parameter-free} & & \text{not closed} & & & & \\ \text{closed} & & & & & & \end{array}$$

Definition (Henkin axioms)

Let $\mathcal{T}$ be consistent and parameter-free. Then define $\mathcal{H}$ as follows:

$$\mathcal{H}_0 = \mathcal{T} \qquad \mathcal{H}_{n+1} = \begin{cases} \mathcal{H}_n \cup \{\varphi_p^x \dot{\rightarrow} \dot{\forall} x.\varphi\} & \text{if } \mathcal{E}_{\mathbf{F}}\ n = \dot{\forall} x.\varphi \\ \quad \text{with } p \text{ fresh in } \mathcal{H}_n & \\ \mathcal{H}_n & \text{otherwise} \end{cases}$$

$$\mathcal{H} := \bigcup \mathcal{H}_n$$

Lemma (Henkin correctness)

- *$\mathcal{H}$ is consistent*
- $(\forall t : \mathcal{T}^c.\ \mathcal{H} \vdash \varphi_t^x) \ \leftrightarrow\ \mathcal{H} \vdash \dot{\forall} x.\ \varphi$

Proof Outline

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\text{Model Existence}} & \vDash_\Omega \mathcal{T} \\ \text{consistent} & & \\ \text{parameter-free} & & \\ \text{closed} & & \end{array}$$

$$\Downarrow$$

$$A \vDash \varphi \to A \vdash \varphi$$

Theorem (Strong quasi-completeness)

Let both $\mathcal{T}$ and φ be closed and parameter-free.

$$\mathcal{T} \vDash \varphi \to \neg\neg \mathcal{T} \vdash \varphi$$

Theorem (Refutation completeness)

$$\mathcal{T} \vdash \varphi \leftrightarrow \mathcal{T} \cup \{\dot\neg \varphi\} \vdash \dot\bot$$

Theorem (Strong quasi-completeness)

Let both $\mathcal{T}$ and φ be closed and parameter-free.

$$\mathcal{T} \vDash \varphi \to \neg\neg\mathcal{T} \vdash \varphi$$

Definition (Stability of $\vdash$)

$$\neg\neg\, A \vdash \varphi \to A \vdash \varphi$$

Theorem (Completeness)

Assume the stability of $\vdash$. Let A and φ be closed and parameter-free.

$$A \vDash \varphi \to A \vdash \varphi$$

Future Work

- Establish Soundness and use AutoSubst
- Completeness of an intuitionistic Gentzen system
- Cut free completeness of intuitionistic ND
- Multiple possibilities:
  - Cut elimination for classical ND
  - Game semantics
  - ...

References

- Hugo Herbelin and Danko Ilik. *An analysis of the constructive content of Henkin's proof of Gödel's completeness theorem*. Draft. 2016.
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- Yannik Forster, Dominik Kirst, and Gert Smolka. *On Synthetic Undecidability in Coq, with an Application to the Entscheidungsproblem*. CPP'19. 2018.