

An Almost Constructive Proof of Classical First-Order Completeness

First Bachelor Seminar Talk

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14th December 2018

Partial History of First-Order Completeness

1928	First formal statement by Hilbert and Ackermann ¹
1929	First proven by Gödel ²
1947	Greatly simplified by Henkin ³
⋮	
2016	Constructive analysis by Herbelin and Ilik ⁴

¹Ackermann and Hilbert. "Grundzüge der theoretischen Logik"

²Gödel. "Über die Vollständigkeit des Logikkalküls"

³Henkin. "The Completeness of the First-Order Functional Calculus"

⁴Herbelin and Ilik. *An analysis of the constructive content of Henkin's proof of Gödel's completeness theorem*

Definition (Syntax)

$$s, t : \mathbf{T} ::= e \mid f \ t \mid x \mid p \qquad x, p : \mathbf{N}$$

$$\varphi, \psi : \mathbf{F} ::= \perp \mid P \ s \ t \mid \varphi \dot{\rightarrow} \psi \mid \dot{\forall} x. \varphi \qquad x : \mathbf{N}$$

$$\dot{\neg} \varphi := \varphi \dot{\rightarrow} \perp \qquad \dot{\exists} x. \varphi := \dot{\neg} \dot{\forall} x. \dot{\neg} \varphi \qquad \varphi \dot{\forall} \psi := \dot{\neg} \varphi \dot{\rightarrow} \psi$$

Definition (Deduction system)

$$\text{CTX} \frac{\varphi \in A}{A \vdash \varphi}$$

$$\text{II} \frac{\varphi :: A \vdash \psi}{A \vdash \varphi \dot{\rightarrow} \psi}$$

$$\text{IE} \frac{A \vdash \varphi \dot{\rightarrow} \psi \quad A \vdash \varphi}{A \vdash \psi}$$

$$\text{DN} \frac{A \vdash \dot{\rightarrow} \dot{\rightarrow} \varphi}{A \vdash \varphi}$$

$$\text{ALLI} \frac{A \vdash \varphi_p^x \quad p \text{ fresh for } \varphi \text{ and } A}{A \vdash \dot{\forall} x. \varphi}$$

$$\text{ALLE} \frac{A \vdash \dot{\forall} x. \varphi \quad t \text{ closed}}{A \vdash \varphi_t^x}$$

Definition (Interpretation)

An interpretation \mathbf{I} on a domain \mathbf{D} consists of:

$$e^{\mathbf{I}} : \mathbf{D} \quad f^{\mathbf{I}} : \mathbf{D} \rightarrow \mathbf{D} \quad \cdot^{\mathbf{I}} : \mathbf{N} \rightarrow \mathbf{D} \quad P^{\mathbf{I}} : \mathbf{D} \rightarrow \mathbf{D} \rightarrow \mathbf{P}$$

Definition (Evaluation)

Given $\rho : \mathbf{N} \rightarrow \mathbf{D}$, we extend \mathbf{I} to $t^{\mathbf{I},\rho} : \mathbf{D}$ and $\rho \vDash_{\mathbf{I}} \varphi : \mathbf{P}$:

$$\begin{aligned} \rho \vDash_{\mathbf{I}} \perp &= \perp \\ \rho \vDash_{\mathbf{I}} P s t &= P^{\mathbf{I}} s^{\mathbf{I},\rho} t^{\mathbf{I},\rho} \\ \rho \vDash_{\mathbf{I}} \varphi \dot{\rightarrow} \psi &= \rho \vDash_{\mathbf{I}} \varphi \rightarrow \rho \vDash_{\mathbf{I}} \psi \\ \rho \vDash_{\mathbf{I}} \dot{\forall} x. \varphi &= \forall d : \mathbf{D}. \rho[x \mapsto d] \vDash_{\mathbf{I}} \varphi \end{aligned}$$

$$A \vDash \varphi := \forall \mathbf{I} \rho. \rho \vDash_{\mathbf{I}} A \rightarrow \rho \vDash_{\mathbf{I}} \varphi$$

Definition (Theories)

We extend the previous notions to theories $\mathcal{T} : \mathbf{F} \rightarrow \mathbf{P}$:

$$\mathcal{T} \models \varphi := \forall \mathbf{I} \rho. \rho \models_{\mathbf{I}} \mathcal{T} \rightarrow \rho \models_{\mathbf{I}} \varphi$$

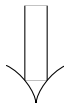
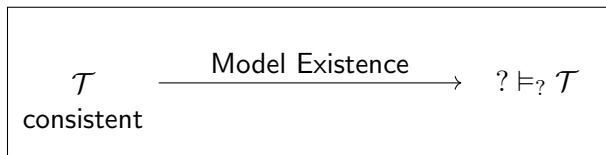
$$\mathcal{T} \vdash \varphi := A \vdash \varphi \exists A. A \subseteq \mathcal{T} \wedge A \vdash \varphi$$

Definition (Consistency)

We call $\mathcal{T} : \mathbf{F} \rightarrow \mathbf{P}$

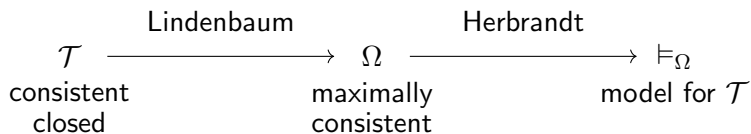
- consistent if $\mathcal{T} \not\vdash \perp$
- maximally consistent if $\mathcal{T} \not\vdash \perp$ and $\varphi \in \mathcal{T}$ if $\mathcal{T} \cup \{\varphi\} \not\vdash \perp$

Proof Outline



$$A \models \varphi \rightarrow A \vdash \varphi$$

Quantifier-free Model Existence



Definition

Given a consistent \mathcal{T} , we fix an enumeration \mathcal{E}_F and define

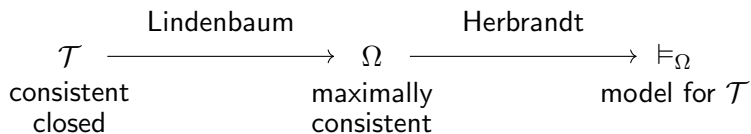
$$\Omega_0 = \mathcal{T} \quad \Omega_{n+1} = \begin{cases} \Omega_n \cup \{\mathcal{E}_F n\} & \Omega_n \cup \{\mathcal{E}_F n\} \text{ consistent} \\ \Omega_n & \text{otherwise} \end{cases}$$

$$\Omega := \bigcup \Omega_n$$

Lemma (Lindenbaum)

Ω is a maximally consistent extension of \mathcal{T} .

Quantifier-free Model Existence



Definition (Herbrandt model)

Given a theory Ω we define its Herbrandt model on closed terms \mathbb{T}^c :

$$t^{\Omega, \rho} := t \qquad P^{\Omega} s t := P s t \in \Omega$$

Lemma (Model correctness)

Let Ω be maximally consistent and φ be closed and quantifier-free, then

$$\vDash_{\Omega} \varphi \leftrightarrow \varphi \in \Omega$$

Corollary (Model existence)

Let \mathcal{T} be consistent and closed, then $\vDash_{\Omega} \mathcal{T}$.

Lemma (Maximally consistent membership)

Let Ω be maximally consistent. Then $\varphi \in \Omega \leftrightarrow \Omega \vdash \varphi$.

Lemma (Model correctness)

Let Ω be maximally consistent and φ be closed and quantifier-free, then

$$\models_{\Omega} \varphi \leftrightarrow \varphi \in \Omega$$

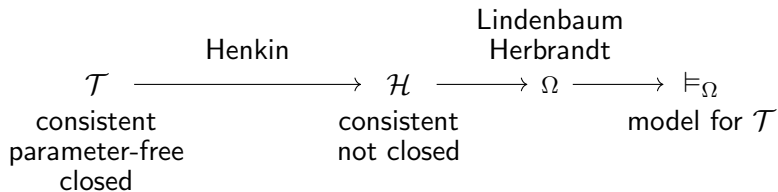
Proof.

Proof per induction on the size of φ . There are three cases:

- $Pst \in \Omega \leftrightarrow Pst \in \Omega$
- $\perp \leftrightarrow \Omega \vdash \perp$
- $(\Omega \vdash \varphi \rightarrow \Omega \vdash \psi) \leftrightarrow \Omega \vdash \varphi \dot{\rightarrow} \psi$



First-Order Model Existence



Definition (Henkin axioms)

Let \mathcal{T} be consistent and parameter-free. Then define \mathcal{H} as follows:

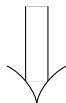
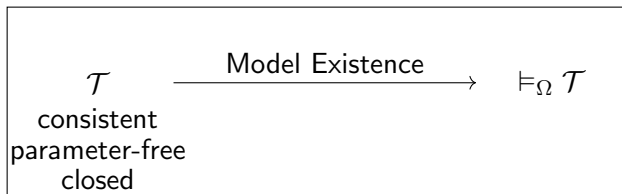
$$\mathcal{H}_0 = \mathcal{T} \quad \mathcal{H}_{n+1} = \begin{cases} \mathcal{H}_n \cup \{\varphi_p^x \dot{\rightarrow} \dot{\forall}x.\varphi\} & \text{if } \mathcal{E}_F n = \dot{\forall}x.\varphi \\ \text{with } p \text{ fresh in } \mathcal{H}_n & \\ \mathcal{H}_n & \text{otherwise} \end{cases}$$

$$\mathcal{H} := \bigcup \mathcal{H}_n$$

Lemma (Henkin correctness)

- \mathcal{H} is consistent
- $(\forall t : T^c. \mathcal{H} \vdash \varphi_t^x) \leftrightarrow \mathcal{H} \vdash \dot{\forall}x.\varphi$

Proof Outline



$$A \models \varphi \rightarrow A \vdash \varphi$$

Theorem (Strong quasi-completeness)

Let both \mathcal{T} and φ be closed and parameter-free.

$$\mathcal{T} \models \varphi \rightarrow \neg\neg\mathcal{T} \vdash \varphi$$

Theorem (Refutation completeness)

$$\mathcal{T} \vdash \varphi \leftrightarrow \mathcal{T} \cup \{\neg\varphi\} \vdash \perp$$

Theorem (Strong quasi-completeness)

Let both \mathcal{T} and φ be closed and parameter-free.

$$\mathcal{T} \models \varphi \rightarrow \neg\neg\mathcal{T} \vdash \varphi$$

Definition (Stability of \vdash)

$$\neg\neg A \vdash \varphi \rightarrow A \vdash \varphi$$

Theorem (Completeness)

Assume the stability of \vdash . Let A and φ be closed and parameter-free.

$$A \models \varphi \rightarrow A \vdash \varphi$$

Future Work

Establish Soundness and use AutoSubst

Completeness of an intuitionistic Gentzen system

Cut free completeness of intuitionistic ND





Multiple possibilities:

- Cut elimination for classical ND

- Game semantics

- ...

References

-  Hugo Herbelin and Danko Ilik. *An analysis of the constructive content of Henkin's proof of Gödel's completeness theorem*. Draft. 2016.
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-  Melvin Fitting. *First-Order Logic and Automated Theorem Proving*. Springer, 1996.
-  Yannik Forster, Dominik Kirst, and Gert Smolka. *On Synthetic Undecidability in Coq, with an Application to the Entscheidungsproblem*. CPP'19. 2018.