Soundness and Completeness of Intuitionistic Dialogues

Second Bachelor Seminar Talk

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Recap

$$\boxed{\mathcal{T} \nvdash \bot \xrightarrow{\text{Model existence}} \mathcal{M} \vDash \mathcal{T}}$$

$$\Big\Downarrow \text{Markov's principle}$$

Completeness

A constructive proof

Definition (Tarski Semantics)

Given $\rho : \mathbf{N} \to \mathbf{D}$, we extend classical $\mathbf{I}$ to $\rho \vDash_{\mathbf{I}} \varphi \ : \ \mathbf{P}$:

$$\begin{aligned} \rho \vDash_{\mathbf{I}} \dot{\bot} &= Q \\ \rho \vDash_{\mathbf{I}} P\ s\ t &= P^{\mathbf{I}}\ s^{\mathbf{I},\rho}\ t^{\mathbf{I},\rho} \\ \rho \vDash_{\mathbf{I}} \varphi \dot{\to} \psi &= \rho \vDash_{\mathbf{I}} \varphi \to \rho \vDash_{\mathbf{I}} \psi \\ \rho \vDash_{\mathbf{I}} \dot{\forall} x.\varphi &= \forall d : \mathbf{D}.\ \rho[x \mapsto d] \vDash_{\mathbf{I}} \varphi \end{aligned}$$

$$A \vDash \varphi \ := \ \forall \mathbf{I}\ \rho.\ \rho \vDash_{\mathbf{I}} A \to \rho \vDash_{\mathbf{I}} \varphi$$

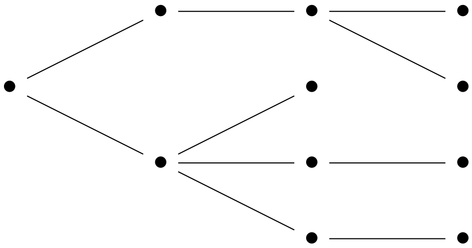
A constructive proof

$$\boxed{\mathcal{T} \nvdash \bot \xrightarrow{\quad\text{Model existence}\quad} \mathcal{M} \models \mathcal{T}}$$

$$\Downarrow$$

Completeness

Kripke models



$$\mathcal{K} = (\mathbf{I}, \mathbf{W}, \preccurlyeq, P_u)$$
$$\forall\, u \preccurlyeq v.\ P_u \subseteq P_v$$

Universal Kripke model[1]

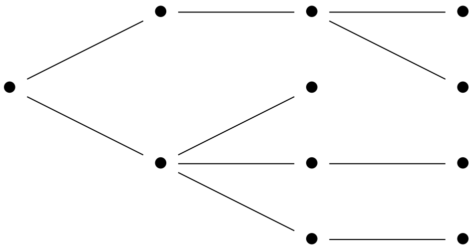
$$\begin{tikzcd}[column sep=small]
 & & & \Gamma, \varphi, \tau & \cdots \\
 & \Gamma, \varphi \arrow[urr, dash] \arrow[drr, dash] & & & \\
\Gamma \arrow[ur, dash] \arrow[dr, dash] & & & \Gamma, \varphi, \sigma & \cdots \\
 & \Gamma, \psi & \cdots & &
\end{tikzcd}$$

$$\mathcal{K} = (\mathbf{l}, \mathcal{L}(\mathtt{F}), \subseteq, \lambda\,\Gamma s t.\ \Gamma \vdash P\ s\ t)$$

$$\mathsf{MP} \to \rho \vDash_\Gamma \varphi \to \Gamma \vdash \varphi[\rho]$$

[1] Herbelin and Lee. "Forcing-based cut-elimination for gentzen-style intuitionistic sequent calculus"

A constructive proof



$$\mathcal{K} = (\mathbf{I}, \mathbf{W}, \preccurlyeq, P_u, \bot_u)$$

Partial History of Dialogue Semantics

1958 — Lorenzen describes material dialogues [2]

1961 — Lorenz formalizes dialogues as games [3]

1985 — Felscher gives a rigorous completeness proof [4]

2007 — Sørensen and Urzyczyn give a generic completeness proof [5]

[2] Lorenzen. "Logik und Agon"

[3] Lorenz. "Arithmetik und Logik als Spiele"

[4] Felscher. "Dialogues, strategies, and intuitionistic provability"

[5] Sørensen and Urzyczyn. "Sequent calculus, dialogues, and cut elimination"

Attacks & Defenses

$$\lhd : \mathtt{F} \to \mathcal{A} \to \mathcal{O}(\mathtt{F}) \to \mathbf{P}$$
$$\mathcal{D}. : \mathcal{A} \to (\mathtt{F} \to \mathbf{P})$$

Attacks	$\mathcal{D}_a$
$\bot \lhd A_\bot$	—
$\varphi \to \psi \lhd A_\to \mid \ulcorner \varphi \urcorner$	$\{\psi\}$
$\varphi \lor \psi \lhd A_\lor$	$\{\varphi, \psi\}$
$\varphi \land \psi \lhd A_L$	$\{\varphi\}$
$\varphi \land \psi \lhd A_R$	$\{\psi\}$
$\forall \varphi \lhd A_t$	$\{\varphi[t]\}$
$\exists \varphi \lhd A_\exists$	$\{\varphi[t] \mid t : \mathtt{T}\}$

$$\varphi \lhd a := \varphi \lhd a \mid \emptyset$$

Dialogues

$$\underline{(P(x) \to Q(x)) \to P(x) \to P(x) \wedge Q(x)}$$

"Let's assume $P(x) \to Q(x)$."	O: $P(x) \to Q(x)$
"Then $P(x) \to P(x) \wedge Q(x)$."	P: $P(x) \to P(x) \wedge Q(x)$
"Assuming $P(x)$, $P(x) \wedge Q(x)$ follows?"	O: $A_{\to}\ P(x)$
"Yes."	P: $P(x) \wedge Q(x)$
"So $Q(x)$ holds?"	O: A_R
"As $P(x) \to Q(x)$, $Q(x)$ holds?"	P: $A_{\to}\ P(x)$
"Yes."	O: $Q(x)$
"Then $Q(x)$ holds."	P: $Q(x)$

Structure of dialogues

- Two player game
- Opponent makes admissions
- Proponent makes claim
- Players take turns, either attack or defend

O: $P(x) \to Q(x)$

P: $P(x) \to P(x) \wedge Q(x)$

O: $A_{\to}\ P(x)$

P: $P(x) \wedge Q(x)$

O: A_R

P: $A_{\to}\ P(x)$

O: $Q(x)$

P: $Q(x)$

Structure of dialogues

- Opponent reacts to previous move
- Proponent may attack any admission
- Proponent may defend against the last attack
- Proponent may only admit atomic formulas after the opponent has done so
- A dialogue is won if the opponent can't react

O: $P(x) \to Q(x)$

P: $P(x) \to P(x) \land Q(x)$

O: $A_{\to}\ P(x)$

P: $P(x) \land Q(x)$

O: A_R

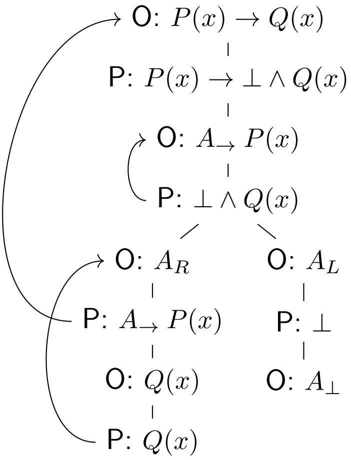
P: $A_{\to}\ P(x)$

O: $Q(x)$

P: $Q(x)$

Winning & Validity

$$\underline{(P(x) \to Q(x)) \to P(x) \to \bot \wedge Q(x)}$$



Formalizing Dialogues

$$\mathcal{L}(\texttt{F}) \times \mathcal{A}$$

$$\mathcal{M} := PA\,(a : \mathcal{A}) \mid PD\,(\varphi : \texttt{F})$$

$$\rightsquigarrow_p \,: \mathcal{S} \to \mathcal{M} \to \mathbf{P}$$

$$\rightsquigarrow_o \,: \mathcal{S} \to \mathcal{M} \to \mathcal{S} \to \mathbf{P}$$

Proponent moves

Proponent may attack any admission

$$\frac{\varphi \in A_o \quad \varphi \lhd a \,|\, \psi \quad \text{justified } A_o\, \psi}{(A_o, c) \leadsto_p P A\, a}$$

Proponent may defend against the last attack

$$\frac{\varphi \in \mathcal{D}_c \quad \text{justified } A_o\, \varphi}{(A_o, c) \leadsto_p P D\, \varphi}$$

Proponent may only admit atomic formulas after the opponent has done so

$$\text{justified } A_o\, \varphi := \varphi \in \mathtt{F}^a \to \varphi \in A_o$$

Opponent moves

Opponent may attack preceding defense

$$\frac{\varphi \lhd c' \mid \psi}{(A_o, c) \,;\, PD\, \varphi \rightsquigarrow_o (\psi :: A_o, c')}$$

Opponent may defend against preceding attack

$$\frac{\varphi \in \mathcal{D}_a}{(A_o, c) \,;\, PA\, a \rightsquigarrow_o (\varphi :: A_o, c)}$$

Opponent may counter preceding attack

$$\frac{\varphi \lhd a \mid \ulcorner\psi\urcorner \qquad \psi \lhd c' \mid \tau}{(A_o, c) \,;\, PA\, a \rightsquigarrow_o (\tau :: A_o, c')}$$

Winning & Validity

$$\frac{s \rightsquigarrow_p m \quad \forall s'.\ s\,;m \rightsquigarrow_o s' \to \mathsf{Win}\ s'}{\mathsf{Win}\ s}$$

$$\Gamma \vDash \varphi := \forall\, \varphi \lhd c \mid \psi.\ \mathsf{Win}\ (\psi :: \Gamma, c)$$

Sequent Calculus LJD

$$\vdash : \mathcal{L}(\mathtt{F}) \to (\mathtt{F} \to \mathbf{P}) \to \mathbf{P}$$

$$\text{L}\,\frac{\begin{array}{c}\varphi \in \Gamma \quad \varphi \lhd a \mid \psi \\ \textsf{justified}\ \Gamma\,\psi \quad \forall \sigma \in \mathcal{D}_a.\ \Gamma, \sigma \vdash \Delta \quad \forall \psi \lhd a' \mid \tau.\ \Gamma, \tau \vdash \mathcal{D}_{a'}\end{array}}{\Gamma \vdash \Delta}$$

$$\text{R}\,\frac{\varphi \in \Delta \quad \textsf{justified}\ \Gamma\,\varphi \quad \forall \varphi \lhd a \mid \psi.\ \Gamma, \psi \vdash \mathcal{D}_a}{\Gamma \vdash \Delta}$$

Soundness & Completeness

Theorem

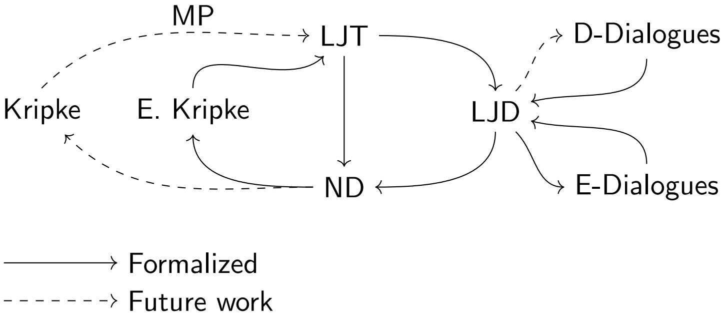
- $\Gamma \vdash \{\varphi\} \rightarrow \Gamma \vDash \varphi$
- $\Gamma \vDash \varphi \rightarrow \Gamma \vdash \{\varphi\}$

Proof.

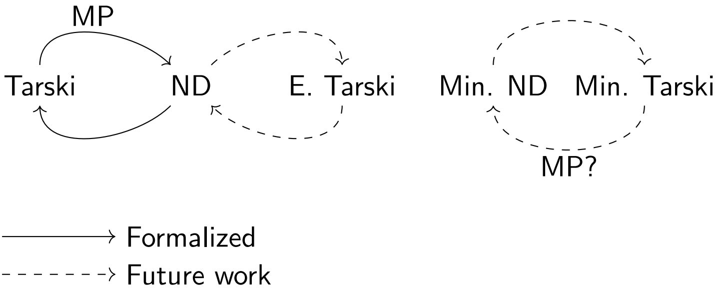
- Show $\forall \Gamma, \Delta.\ \Gamma \vdash \Delta \rightarrow \forall c.\ \Delta \subseteq \mathcal{D}_c \rightarrow \text{Win}\ (\Gamma, c)$.
- Show $\forall A_o, c.\ \text{Win}\ (A_o, c) \rightarrow A_o \vdash \mathcal{D}_c$.

$\square$

Intuitionistic results ($\forall$,$\to$,$\perp$-fragment)



Classical results



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