SEMANTICS OF INTUITIONISTIC PROPOSITIONAL LOGIC: HEYTING ALGEBRAS AND KRIPKE MODELS

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Abstract. We describe two well-known semantics for intuitionistic propositional logic: Heyting algebras and Kripke models. We prove both semantics are sound for an intuitionistic propositional logic with only implication and false. We use Heyting algebras to prove undefinability results. We also prove a Kripke model can be converted into a Heyting algebra. If the metalogic is classical we can convert a Heyting algebra into a Kripke model. In some ways our task is made easier by the fact that the logic does not have disjunction. The results have been formalized in Coq.

1. Introduction

We study two well-known semantics of intuitionistic propositional logic: Heyting algebras and Kripke models. To simplify matters we consider propositional logic with only implication and false. For some results the fact that we do not include disjunction is a significant simplification. Most of these results can be found in Chapter 2 of [4]. Many historical notes and other references can be found starting from [4].

The results have been formalized in Coq [2] and we will at times give pointers to the formalization.

2. Intuitionistic Propositional Logic

We use s, t to range over propositional formulas, which we take to be defined by the grammar

$$x|s \to t|\bot$$

where x ranges over (propositional) variables. For finite lists Γ of propositional formulas, $\Gamma \vdash s$ is defined using the natural deduction calculus in Figure 1 (see [3]).

3. Heyting Algebras

Heyting algebras originated by using topological spaces to give models of intuitionistic logic. According to [4] the technique can be traced back to Tarski [5].

A Heyting algebra is typically defined as a lattice with an implication operation. In the formalization we did not need antisymmetry, so we dropped this condition. One could argue that the definition is of a "Heyting prealgebra," but we will simply say Heyting algebra.

In Coq one can represent the collection of Heyting algebras as a record type consisting of the following information:[1] a type H, a relation $\leq$, $\bot : H$, $\top : H$, operations

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[1] See `HeytingAlgebra` in the formalization.

$$\frac{}{\Gamma \vdash s}\ s \in \Gamma \qquad \frac{\Gamma, s \vdash t}{\Gamma \vdash s \to t} \qquad \frac{\Gamma \vdash s \to t \qquad \Gamma \vdash s}{\Gamma \vdash t} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash s}$$

FIGURE 1. Natural Deduction Rules

$\wedge, \vee, \Rightarrow$: $H \to H \to H$ and a number of properties.[2] The properties are that $\leq$ is reflexive and transitive (a partial preorder), $\bot$ is $\leq$-least, $\top$ is $\leq$-greatest, $\wedge$ gives greatest lower bounds, $\vee$ gives least upper bounds, and that $u \Rightarrow v$ is the greatest element w such that $w \wedge u \leq v$.

Given a Heyting algebra A, an interpretation φ (into A) maps variables to H_A. Given a Heyting algebra A and an interpretation φ, we can define an evaluation function $[\![-]\!]^A_\varphi$ mapping formulas to elements of H_A by recursion:

$$[\![x]\!]^A_\varphi = \varphi(x)$$

$$[\![s \to t]\!]^A_\varphi = [\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi$$

and

$$[\![\bot]\!]^A_\varphi = \bot.$$

If we included conjunction and/or disjunction in the propositional logic, then we would interpret them using $\wedge$ and $\vee$ in the Heyting algebra.

We can also extend the evaluation function to lists Γ of propositions using $\top$ and $\wedge$:

$$[\![\cdot]\!]^A_\varphi = \top$$

and

$$[\![\Gamma, s]\!]^A_\varphi = [\![\Gamma]\!]^A_\varphi \wedge [\![s]\!]^A_\varphi.$$

We can now state and prove soundness.[3]

Theorem 3.1. *If* $\Gamma \vdash s$, *then* $[\![\Gamma]\!]^A_\varphi \leq [\![\Gamma]\!]^s_\varphi$

Proof. We argue by induction on the derivation of $\Gamma \vdash s$. If $s \in \Gamma$, then $[\![\Gamma]\!]^A_\varphi \leq [\![\Gamma]\!]^s_\varphi$ follows by an easy subinduction on Γ using transitivity of $\leq$ and the properties of $\wedge$.

For the implication introduction rule, the inductive hypothesis gives

$$[\![\Gamma]\!]^A_\varphi \wedge [\![s]\!]^A_\varphi \leq [\![t]\!]^A_\varphi.$$

Since $[\![\Gamma]\!]^A_\varphi$ is an element w such that

$$w \wedge [\![s]\!]^A_\varphi \leq [\![t]\!]^A_\varphi$$

and $[\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi$ is the greatest such element, we conclude

$$[\![\Gamma]\!]^A_\varphi \leq [\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi$$

which is precisely what we need to prove.

For the implication elimination rule, the inductive hypotheses give

$$[\![\Gamma]\!]^A_\varphi \leq [\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi$$

[2] We use $\bot$ both for a propositional formula and the bottom element of a Heyting algebra, since confusion seems unlikely.

[3] See `nd_soundH` in the formalization.

and

$$[\![\Gamma]\!]^A_\varphi \leq [\![s]\!]^A_\varphi.$$

Since $\wedge$ gives greatest upper bounds, we know

$$[\![\Gamma]\!]^A_\varphi \leq ([\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi) \wedge [\![s]\!]^A_\varphi$$

and so

$$[\![\Gamma]\!]^A_\varphi \leq [\![t]\!]^A_\varphi$$

by transitivity of $\leq$ and the main property of $\Rightarrow$.

For the false elimination rule, the inductive hypothesis gives

$$[\![\Gamma]\!]^A_\varphi \leq \perp.$$

Since $\perp$ is $\leq$-least, we know $[\![\Gamma]\!]^A_\varphi$ is also $\leq$-least. In particular, $[\![\Gamma]\!]^A_\varphi \leq [\![s]\!]^A_\varphi$. □

4. Examples of Heyting Algebras

We consider two examples of Heyting Algebras. Both will consist of 5 elements. The first will demonstrate that conjunction cannot be expressed via implication and false. The second will demonstrate that disjunction cannot be expressed via implication and false.

Example 4.1. *Let H be $\{\perp, c, a, b, \top\}$.*[4] *We define $\leq$ so that $\leq$ is reflexive, $\perp$ is the only $\leq$-least element, $\top$ is the only $\leq$-greatest element, and $a \not\leq b$, $b \not\leq a$, $c \leq a$ but $a \not\leq c$, $c \leq b$ but $b \not\leq c$. We give $\wedge$, $\vee$ and $\Rightarrow$ using tables.*

$\wedge$	$\perp$	c	a	b	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
c	$\perp$	c	c	c	c
a	$\perp$	c	a	c	a
b	$\perp$	c	c	b	b
$\top$	$\perp$	c	a	b	$\top$

$\vee$	$\perp$	c	a	b	$\top$
$\perp$	$\perp$	c	a	b	$\top$
c	c	c	a	b	$\top$
a	a	a	a	$\top$	$\top$
b	b	b	$\top$	b	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\Rightarrow$	$\perp$	c	a	b	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$
c	$\perp$	$\top$	$\top$	$\top$	$\top$
a	$\perp$	b	$\top$	b	$\top$
b	$\perp$	a	a	$\top$	$\top$
$\top$	$\perp$	c	a	b	$\top$

This is chosen so that c is $a \wedge b$ and is not $\perp$. It is easy to check this is a Heyting algebra.[5] *Let A be this Heyting algebra. As an interpretation, consider φ such that $\varphi(x) = a$ for a chosen variable x and $\varphi(y) = b$ for all other variables y. An easy induction on s can be used to prove $[\![s]\!]^A_\varphi \neq c$ for all s. Clearly if we had conjunction in our logic this would be false, since $[\![x]\!]^A_\varphi \wedge [\![y]\!]^A_\varphi = a \wedge b = c$. Hence conjunction is not definable using implication and false.*

For the next example we consider a Heyting algebra dual to the Heyting algebra in the previous example.

[4]In the formalization this is the inductive type `Ha5` with five elements `Ha5bot` for $\perp$, `Ha5ab` for c, `Ha5a` for a, `Ha5b` for b and `Ha5top` for $\top$.

[5]In fact, since it is computational, Coq can check each of the conditions simply by case analysis, simplification and `tauto`.

Example 4.2. *Let H be $\{\bot, a, b, c, \top\}$.*[6] *We define $\leq$ so that $\leq$ is reflexive, $\bot$ is the only $\leq$-least element, $\top$ is the only $\leq$-greatest element, and $a \not\leq b$, $b \not\leq a$, $a \leq c$ but $c \not\leq a$, $b \leq c$ but $c \not\leq b$. We define the operations as follows.*

$\wedge$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a	a
b	$\bot$	$\bot$	b	b	b
c	$\bot$	a	b	c	c
$\top$	$\bot$	a	b	c	$\top$

$\vee$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	a	b	c	$\top$
a	a	a	c	c	$\top$
b	b	c	b	c	$\top$
c	c	c	c	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\Rightarrow$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	$\top$	$\top$
b	a	a	$\top$	$\top$	$\top$
c	$\bot$	a	b	$\top$	$\top$
$\top$	$\bot$	a	b	c	$\top$

Again it is easy to check this is a Heyting algebra, which we again call A. We again take the intepretation such that $\varphi(x) = a$ for a chosen x and $\varphi(y) = b$ for all other variables y. Again we can prove $[\![s]\!]^A_\varphi \neq c$ for all s. In this case we can conclude that disjunction is not definable using implication and false since $[\![x]\!]^A_\varphi \vee [\![y]\!]^A_\varphi = c$.

5. Kripke Models

Kripke models were introduced by Kripke in 1963 [1]. A Kripke model is a partial order of "states" (or "worlds") with a relation indicating which variables are true at each state. Again, we will not need antisymmetry, so we will omit it.

In Coq we represent Kripke models as a record type consisting of the following information:[7] a type of states, a $\leq$ relation which is reflexive and transitive, and a labelling relation L on variables and states such that if $L(x, \sigma)$ and $\sigma \leq \tau$, then $L(x, \tau)$.

Given a Kripke model M, we define an evaluation function $[\![-]\!]^M$ taking propositional formulas to sets of states as follows:[8]

$$[\![x]\!]^M = \{\sigma | L_M(x, \sigma)\}$$

$$[\![s \to t]\!]^M = \{\sigma | \forall \tau. \sigma \leq \tau \to \tau \in [\![s]\!]^M \to \tau \in [\![t]\!]^M\}$$

$$[\![\bot]\!]^M = \emptyset$$

For lists Γ, $[\![\Gamma]\!]^M = \{\sigma | \forall s \in \Gamma. \sigma \in [\![s]\!]^M\}$.

We prove the following monotonicity result. If we included conjunction or disjunction in our logic, the result would require induction on formulas. Since we only have implication and false, a case analysis suffices.[9]

Lemma 5.1. *If $\sigma \leq \tau$ and $\sigma \in [\![s]\!]^M$, then $\tau \in [\![s]\!]^M$. In other words, each $[\![s]\!]^M$ is upwards closed.*

Proof. For variables we assumed this property for L. For $\bot$ the result is trival since $[\![\bot]\!]^M$ is empty. For $s \to t$, suppose $\sigma \leq \tau$ and $\sigma \in [\![s \to t]\!]^M$. We need to prove $\tau \in [\![s \to t]\!]^M$. Let μ be a state such that be such that $\tau \leq \mu$ and $\mu \in [\![s]\!]^M$. Since $\leq$ is transitive, $\sigma \leq \mu$. Hence $\mu \in [\![t]\!]^M$ follows from $\sigma \in [\![s \to t]\!]^M$ and we are done. □

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[6] In the formalization this is the inductive type `Hb5` with five elements `Hb5bot` for $\bot$, `Hb5a` for a, `Hb5b` for b, `Hb5ab` for c and `Hb5top` for $\top$.

[7] See `KripkeModel` in the formalization.

[8] See `evalK` in the formalization.

[9] See `monotone` in the formalization.

We can now prove the following soundness result.[10]

Theorem 5.1. *If* $\Gamma \vdash s$ *and* $\sigma \in [\![\Gamma]\!]^M$, *then* $\sigma \in [\![s]\!]^M$.

Proof. We argue by induction on the derivation of $\Gamma \vdash s$. The assumption case is trivial.

For the implication introduction rule, assume we have $\sigma \in [\![\Gamma]\!]^M$. We need to prove $\sigma \in [\![s \to t]\!]^M$. Let τ be such that $\sigma \leq \tau$ and $\tau \in [\![s]\!]^M$. By monotonicity, we know $\tau \in [\![\Gamma]\!]^M$ and so $\tau \in [\![\Gamma, s]\!]^M$. By the inductive hypothesis we know $\tau \in [\![t]\!]^M$ and we are done.

For the implication elimination rule, assume we have $\sigma \in [\![\Gamma]\!]^M$. The inductive hypotheses give $\sigma \in [\![s \to t]\!]^M$ and $\sigma \in [\![s]\!]^M$. Since $\leq$ is reflexive, we conclude $\sigma \in [\![t]\!]^M$ as desired.

Finally we consider the false elimination rule. Assume we have $\sigma \in [\![\Gamma]\!]^M$. The inductive hypothesis implies $\sigma \in [\![\bot]\!]^M = \emptyset$, a contradiction. ☐

6. Heyting Algebras from Kripke Models

Suppose M is a Kripke model with (pre)ordering $\leq$ and labelling relation L. We can construct a Heyting algebra by consider the upward closed sets relative to $\leq$. That is, we take H to be the sets X of states such that if $\sigma \in X$ and $\sigma \leq \tau$, then $\tau \in X$.[11] For the ordering on H we take $\subseteq$. We take $\bot$ to be the empty set of states and $\top$ to be the set of all states. We define $\wedge$ by intersection and $\vee$ by union as expected. Given $X, Y \in H$, we define $X \Rightarrow Y$ to be the set

$$\{\sigma | \exists Z \in H. Z \cap X \subseteq Y \text{ and } \sigma \in Z\}.$$

The fact that this yields a Heyting algebra is easy to verify.[12] We write A^M for this Heyting algebra.

The Kripke model also yields an interpretation φ^M into the Heyting algebra by $\varphi^M(x) := \{\sigma | L(x, \sigma)\}$.

We can then prove the Kripke model and the corresponding Heyting algebra agree.[13]

Theorem 6.1. $\sigma \in [\![s]\!]^M$ *if and only if* $\sigma \in [\![s]\!]^{A^M}_{\varphi^M}$.

Proof. The proof is by induction on s. For variables and $\bot$ the equivalence is obvious. We turn immediately to the implication case.

Suppose $\sigma \in [\![s \to t]\!]^M$. We need to prove $\sigma \in [\![s \to t]\!]^{A^M}_{\varphi^M}$. By the definition of $\Rightarrow$ above, we need to prove there is some $Z \in H$ such that $Z \cap [\![s]\!]^M \subseteq [\![t]\!]^M$ and $\sigma \in Z$. Let Z be $[\![s \to t]\!]^M$. By Lemma 5.1 Z is upward closed, i.e., $Z \in H$. By reflexivity of $\leq$ we know $[\![s \to t]\!]^M \cap [\![s]\!]^M \subseteq [\![t]\!]^M$. Since $\sigma \in [\![s \to t]\!]^M$ we are done.

For the other direction suppose $\sigma \in [\![s \to t]\!]^{A^M}_{\varphi^M}$. We will prove $\sigma \in [\![s \to t]\!]^M$. Let τ be a state such that $\sigma \leq \tau$ and $\tau \in [\![s]\!]^M$. We will prove $\tau \in [\![t]\!]^M$. Since $\sigma \in [\![s \to t]\!]^{A^M}_{\varphi^M}$ there is some upward closed Z such that $Z \cap [\![s]\!]^M \subseteq [\![t]\!]^M$ and $\sigma \in Z$. Hence $\tau \in Z$ and so $\tau \in [\![t]\!]^M$ as desired. ☐

[10] See `nd_soundK` in the formalization.

[11] In the formalization we use a sigma type.

[12] See `HeytingAlgebraOfKripkeModel` in the formalization.

[13] See `evalK_evalH_agree` in the formalization.

7. Kripke Models from Heyting Algebras

Under enough assumptions one can convert a Heyting algebra into a Kripke model. The argument in [4] makes use of Zorn's Lemma (a form of the axiom of choice). However, since we do not have disjunction we can simplify the construction. As a consequence, we will not need the axiom of choice, but we will still need to assume excluded middle in order to prove the two interpretations agree. We leave open the possibility that the agreement can be proven (for the simplified logic with only implication and false) without using excluded middle.

Suppose A is a Heyting algebra with components H, $\leq$, $\bot$, $\top$, $\wedge$, $\vee$ and $\Rightarrow$. A *proper filter* is a set $F \subseteq H$ such that $\bot \notin F$, $\top \in F$, F is closed under $\wedge$, and F is $\leq$-upwards closed. A *prime filter* is a proper filter F such that if $a \vee b \in F$, then $a \in F$ or $b \in F$.

In [4] a Kripke model is formed by taking prime filters as states. Zorn's Lemma is required to extend a proper filter to an appropriate prime filter. Using prime filters is important for handling disjunction. Since we do not have disjunction, we can simply take proper filters as states and avoid the use of Zorn's Lemma.

Suppose φ is an interpretation into A. We define a Kripke model M^A_φ as follows:[14] The states are proper filters, the order is given by $\subseteq$ and the labelling relation L is given such that $L(x, F)$ holds if $\{a \in H | \varphi(x) \leq a\} \subseteq F$.

The fact that this is a Kripke model is easy to verify and does not require excluded middle.

We finally prove the agreement theorem using excluded middle.[15]

Theorem 7.1. *Let F be a proper filter. $F \in [\![s]\!]^{M^A_\varphi}$ if and only if $[\![s]\!]^A_\varphi \in F$.*

Proof. The proof is by induction on s. It is easy to see that neither $F \in [\![\bot]\!]^{M^A_\varphi}$ nor $[\![\bot]\!]^A_\varphi \in F$ can hold since $\bot \notin F$.

For variables we need to prove $F \in [\![x]\!]^{M^A_\varphi}$ is equivalent to $[\![x]\!]^A_\varphi \in F$. That is, $\{a \in H|\varphi(x) \leq a\} \subseteq F$ if and only if $\varphi(x) \in F$. If $\{a \in H|\varphi(x) \leq a\} \subseteq F$, then $\varphi(x) \in F$ since $\varphi(x) \in \{a \in H|\varphi(x) \leq a\}$. If $\varphi(x) \in F$, $a \in H$ and $\varphi(x) \leq a$, then $a \in F$ since F is upwards closed.

It only remains to consider the implication case. For the first direction assume $F \in [\![s \to t]\!]^{M^A_\varphi}$. We need to prove $[\![s \to t]\!]^A_\varphi \in F$. Let G be $\{a \in H|\exists b \in F.b \wedge [\![s]\!]^A_\varphi \leq a\}$. It is easy to see that $\top \in G$ since $\top \in F$. Likewise G is clearly upward closed: if $a \in G$ and $a \leq a'$, then the same witness $b \in F$ can be used to prove $a' \in G$. Also, G is closed under $\wedge$: If $b \in F$ witnesses $a \in G$ and $b' \in F$ witnesses $a' \in G$, then $b \wedge b' \in F$ witnesses $a \wedge a' \in G$. In order to conclude G is a proper filter, we only need to prove $\bot \notin G$. However, it is possible that $\bot$ actually is a member of G. This is where we need to use excluded middle: either $\bot \in G$ or $\bot \notin G$.

Suppose $\bot \in G$. Then there is some $b \in F$ such that $b \wedge [\![s]\!]^A_\varphi \leq \bot$. Recall we want to prove $[\![s \to t]\!]^A_\varphi \in F$. Since F is upward closed it is enough to prove $b \leq [\![s \to t]\!]^A_\varphi$. Since $b \wedge [\![s]\!]^A_\varphi \leq \bot \leq [\![t]\!]^A_\varphi$ we know $b \leq [\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi$ as desired.

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[14] See `KripkeModelOfHeytingAlgebra` in the formalization.

[15] See `evalH_evalK_agree` in the formalization.

Suppose $\bot \notin G$. In this case G is a proper filter. Also, $F \subseteq G$ since each $a \in F$ can be used as the witness that $a \in G$. We can use $\top \in F$ to witness $[\![s]\!]^A_\varphi \in G$. By the inductive hypothesis for s we know $G \in [\![s]\!]^{M^A_\varphi}$. Since $F \in [\![s \to t]\!]^{M^A_\varphi}$, $F \subseteq G$ and $G \in [\![s]\!]^{M^A_\varphi}$, we know $G \in [\![t]\!]^{M^A_\varphi}$. By the inductive hypothesis for t we know $[\![t]\!]^A_\varphi \in G$. Hence there is a $b \in F$ such that $b \wedge [\![s]\!]^A_\varphi \leq [\![t]\!]^A_\varphi$ and so $b \leq [\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi$. Since F is upwards closed we conclude $[\![s \to t]\!]^A_\varphi \in F$ as desired.

For the other direction assume $[\![s \to t]\!]^A_\varphi \in F$. We need to prove $F \in [\![s \to t]\!]^{M^A_\varphi}$. Let G be a proper filter such that $F \subseteq G$ and $G \in [\![s]\!]^{M^A_\varphi}$. We need to prove $G \in [\![t]\!]^{M^A_\varphi}$. By the inductive hypothesis for s we know $[\![s]\!]^A_\varphi \in G$. Since $F \subseteq G$ we know $[\![s \to t]\!]^A_\varphi \in G$. Note that

$$([\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi) \wedge [\![s]\!]^A_\varphi \leq [\![t]\!]^A_\varphi.$$

Since G is a proper filter we conclude $([\![s]\!]^A_\varphi \Rightarrow [\![t]\!]^A_\varphi) \wedge [\![s]\!]^A_\varphi$ is in G and so $[\![t]\!]^A_\varphi \in G$. By the inductive hypothesis for t we have $G \in [\![t]\!]^{M^A_\varphi}$ as desired. □

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