Terminating Tableaux for Graded Hybrid Logic with Global Modalities and Role Hierarchies

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Abstract. We present a terminating tableau calculus for graded hybrid logic with global modalities, reflexivity, transitivity and role hierarchies. Termination of the system is achieved through pattern-based blocking. Previous approaches to related logics all rely on chain-based blocking. Besides being conceptually simple and suitable for efficient implementation, the pattern-based approach gives us a NExpTime complexity bound for the decision procedure.

1 Introduction

Graded modal logic [1] is a powerful generalization of basic modal logic. Most prominently, graded modalities are used in description logics, rich modal languages tailored for knowledge representation that have a wide range of practical applications [2]. Graded modal logic allows to constrain the number of accessible states satisfying a certain property. So, the modal formula $\Diamond_n p$ is true in a state x if x has at least $n + 1$ successors satisfying p. Analogously to ordinary modal logic, graded modal logic can be extended by nominals [3]. The resulting language, graded hybrid logic, can be extended further by adding global modalities [4], which allow to specify properties that are to hold in all states.

Role hierarchies were first studied by Horrocks [5] in the context of description logics. Using inclusion assertions of the form $r \sqsubseteq r'$, one can specify that the role (relation) r is contained in the role r'. Role hierarchies are of particular interest when considered together with transitivity assertions for roles [6, 7]. The description logic $\mathcal{SHOQ}$ [8] combines the expressive means provided by nominals, graded modalities, role hierarchies and transitive roles.

We present a terminating tableau calculus for graded multimodal logic extended by nominals, global modalities, reflexive and transitive roles, and role hierarchies. The modal language under consideration in the present work is equivalent to $\mathcal{SHOQ}$ extended by reflexive roles and a universal role, both extensions also being known from $\mathcal{SROIQ}$ [9].

The most important difference of our approach to existing calculi for $\mathcal{SHOQ}$ and stronger logics [8, 10, 9] is the technique used to achieve termination of the tableau construction. The established tableau algorithms all rely on modifications of Kripke's chain-based blocking technique [11]. Chain-based blocking assumes a precedence order on the nominals (also known as nodes or prefixes) of a tableau branch, and prevents processing of nominals that are subsumed by preceding nominals. In the simplest case, the precedence order is chosen to be the ancestor relation among nominals (ancestor blocking). In general, however, it may be any order that contains the ancestor relation (anywhere blocking [12, 13]). Ancestor blocking gives an exponential bound on the length of ancestor chains, resulting in a double exponential bound on the size of tableau branches. Depending on the choice of the precedence order, anywhere blocking can lower this bound to a single exponential. However, the size bound on tableau branches does not seem to translate easily to a complexity bound for the decision procedures in [8, 10, 9] ([8, 10] show a 2-NExpTime bound, while [9] leaves complexity open). We feel that the main difficulty in obtaining better complexity bounds is the algorithms being non-cumulative.

A tableau system is called cumulative if its rules never update or delete formulas. In contrast to most systems in the literature, calculi devised for description logics are often not cumulative. Cumulative calculi are easier to present than non-cumulative systems and are usually more amenable to analysis.

Unlike [8, 10, 9], our calculus is cumulative. Cumulativity of the calculus in the presence of nominals is achieved following [14] by representing equality constraints via an equivalence relation on nominals. Termination of our system is achieved through pattern-based blocking [15, 14]. Pattern-based blocking is conceptually simpler than chain-based techniques in that it does not need an order on the nominals, and seems promising as it comes to efficient implementation [16]. Pattern-based blocking provides an exponential bound on the size of tableau branches and on the number of tableau rule applications for a single branch. Thus it limits the complexity of the associated decision procedure to NExpTime. To deal with graded modalities, we extend the blocking conditions in [15, 14], preserving the exponential size bound on the tableau branches.

We begin by presenting a calculus for graded hybrid logic with global modalities. We argue that the blocking conditions used in [15, 14] are insufficient in the presence of graded modalities. We extend pattern-based blocking to account for the increased expressive power and argue the completeness and termination of the resulting calculus. In the second part of the paper, we extend our calculus further by allowing reflexivity, transitivity and inclusion assertions. It turns out that in the presence of inclusion assertions, the blocking condition used for the basic calculus needs to be extended once again.

2 Graded Hybrid Logic with Global Modalities and Role Inclusion

Following [17, 14], we represent modal logic in simple type theory (see [18]). This way we can make use of a rich syntactic and semantic framework and modal logic does not appear as an isolated formal system. We start with two base types B and S. The interpretation of B is fixed and consists of two truth values. The interpretation of S is a nonempty set whose elements are called *worlds* or *states*. Given two types σ and τ, the *functional type* $\sigma\tau$ is interpreted as the set of all total functions from the interpretation of σ to the interpretation of τ. We write $\sigma_1\sigma_2\sigma_3$ for $\sigma_1(\sigma_2\sigma_3)$.

We employ three kinds of variables: *Nominal variables* x, y, z of type S, *propositional variables* p, q of type SB, and *role variables* r of type SSB. Nominal variables are called *nominals* for short, and role variables are called *roles*. We assume there are infinitely many nominals. We use the logical constants

$$\bot, \top : \mathrm{B} \qquad \neg : \mathrm{BB} \qquad \vee, \wedge, \rightarrow : \mathrm{BBB} \qquad \doteq : \mathrm{SSB} \qquad \exists, \forall : (\mathrm{SB})\mathrm{B}$$

Terms are defined as usual. We write st for applications, $\lambda x.s$ for abstractions, and $s_1 s_2 s_3$ for $(s_1 s_2)s_3$. We also use infix notation, e.g., $s \wedge t$ for $(\wedge)st$.

Terms of type B are called *formulas*. We employ some common notational conventions: $\exists x.s$ for $\exists(\lambda x.s)$, $\forall x.s$ for $\forall(\lambda x.s)$, and $x \dot{\neq} y$ for $\neg(x \doteq y)$.

For every $n \in \mathbb{N}$ we define a constant $D_n : \mathrm{S} \ldots \mathrm{SB}$ as follows:

$$D_n \;:=\; \lambda x_1 \ldots \lambda x_n. \bigwedge_{1 \leq i < j \leq n} x_i \dot{\neq} x_j$$

Without loss of generality, we assume a strict total order $\prec$ on the nominals. Given a set of nominals X of cardinality $n \geq 1$, we write DX for $D_n x_1 \ldots x_n$ where $X = \{x_1, \ldots, x_n\}$ and $x_i \prec x_{i+1}$ for $1 \leq i < n$. We write $\bar{D}X$ for $\neg DX$. Formulas of the form DX and $\bar{D}X$ are called *distinctness constraints* on X. Note that for two distinct variables x, y, $\bar{D}\{x, y\}$ reduces to $x \doteq y$.

Moreover, we use the following constants:

$$\begin{array}{ll} \sqsubseteq : (\mathrm{SSB})(\mathrm{SSB})\mathrm{B} & r_1 \sqsubseteq r_2 \;=\; \forall xy. r_1 xy \rightarrow r_2 xy \\ R : (\mathrm{SSB})\mathrm{B} & Rr \;=\; \forall x. rxx \\ T : (\mathrm{SSB})\mathrm{B} & Tr \;=\; \forall xyz. rxy \wedge ryz \rightarrow rxz \end{array}$$

To the right of each constant is an equation defining its semantics. We call formulas of the form $r \sqsubseteq r'$ *(role) inclusion assertions*. Formulas Rr and Tr are called *reflexivity* and *transitivity assertions*, respectively.

We write $\exists X.s$ for $\exists x_1 \ldots x_n.s$ if $|X| = n$ and $X = \{x_1, \ldots, x_n\}$. The *modal constants* are then defined as follows:

$$\begin{array}{ll} \dot{\neg} : (\mathrm{SB})\mathrm{SB} & \dot{\neg} px \;=\; \neg(px) \\ \dot{\wedge} : (\mathrm{SB})(\mathrm{SB})\mathrm{SB} & (p \mathbin{\dot{\wedge}} q)x \;=\; px \wedge qx \\ \dot{\vee} : (\mathrm{SB})(\mathrm{SB})\mathrm{SB} & (p \mathbin{\dot{\vee}} q)x \;=\; px \vee qx \\ \langle \text{-} \rangle_n : (\mathrm{SSB})(\mathrm{SB})\mathrm{SB} & \langle r \rangle_n px \;=\; \exists Y.\, DY \wedge (\bigwedge_{y \in Y} rxy \wedge py) \\ [\text{-}]_n : (\mathrm{SSB})(\mathrm{SB})\mathrm{SB} & [r]_n px \;=\; \forall Y.\, (\bigwedge_{y \in Y} rxy) \rightarrow \bar{D}Y \vee \bigvee_{y \in Y} py \\ E_n : (\mathrm{SB})\mathrm{SB} & E_n px \;=\; \exists Y.\, DY \wedge \bigwedge_{y \in Y} py \\ A_n : (\mathrm{SB})\mathrm{SB} & A_n px \;=\; \forall Y.\, \bar{D}Y \vee \bigvee_{y \in Y} py \\ \dot{\text{-}} : \mathrm{SSB} & \dot{x}y \;=\; x \doteq y \end{array}$$

where $|Y| = n + 1$ in all equations

Intuitively, the semantics of the graded modal operators is as follows:

$E_n p$: There are at least $n+1$ states satisfying p.
$A_n p$: All states but possibly n exceptions satisfy p.
$\langle r\rangle_n p$: There are at least $n+1$ r-successors satisfying p.
$[r]_n p$: All r-successors but possibly n exceptions satisfy p.

In accordance with the usual modal intuition, "formulas" of modal logic are seen as predicates of type SB denoting sets of states. They can be represented as *modal expressions* according to the following grammar:

$$t \ ::= \ p \mid \dot{x} \mid \dot{\neg} t \mid t \mathbin{\dot{\wedge}} t \mid t \mathbin{\dot{\vee}} t \mid \langle r\rangle_n t \mid [r]_n t \mid E_n t \mid A_n t$$

As with the propositional connectives, we use infix notation for $\dot{\wedge}$ and $\dot{\vee}$. Unlike with the propositional connectives, we assume the application of modal operators to have a higher precedence than regular functional application. So, for instance, we write $\dot{\neg}\langle r\rangle_2 \dot{y} \mathbin{\dot{\vee}} p\ x$ for $((\dot{\neg}(\langle r\rangle_2(\dot{y}))) \mathbin{\dot{\vee}} p)x$.

An *interpretation* is a function $\mathfrak{M}$ mapping B to the set $\{0,1\}$, S to a non-empty set, a functional type $\sigma\tau$ to the set of all total functions from the interpretation of σ to the interpretation of τ, interpreting all variables as elements of their respective types, and giving $\bot$, $\top$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, $\forall$, $\doteq$ their usual meaning. A *modal interpretation* is an interpretation that, in addition, satisfies the equations defining the constants $\sqsubseteq$, R, T, $\dot{\neg}$, $\dot{\wedge}$, $\dot{\vee}$, $\langle\_\rangle_n$, $[\_]_n$, E, A, $\dot{\_}$. If $\mathfrak{M}t = 1$, we say that $\mathfrak{M}$ *satisfies* t. A formula is called *satisfiable* if it has a satisfying modal interpretation.

3 Graded Hybrid Logic with Global Modalities

We begin with a tableau calculus for the restricted language without inclusion, reflexivity or transitivity assertions.

3.1 Tableaux and Evidence

For the sake of simplicity, we define our tableau calculus on negation normal expressions, i.e., terms of the form:

$$t \ ::= \ p \mid \dot{\neg} p \mid \dot{x} \mid \dot{\neg}\dot{x} \mid t \mathbin{\dot{\wedge}} t \mid t \mathbin{\dot{\vee}} t \mid \langle r\rangle_n t \mid [r]_n t \mid E_n t \mid A_n t$$

A *branch* Γ is a finite set of formulas s of the form

$$s \ ::= \ tx \mid rxy \mid DX \mid \bar{D}X \mid \bot$$

where t is a negation-normal modal expression of the above form. Formulas of the form rxy are called *accessibility formulas* or *edges*. We use the formula $\bot$ to explicitly mark unsatisfiable branches. We call a branch Γ *closed* if $\bot \in \Gamma$. Otherwise, Γ is called *open*. The branch consisting of the initial formula to be tested for satisfiability is called the *initial branch*.

Let Γ be a branch. With $\sim_\Gamma$ we denote the least equivalence relation $\sim$ on nominals such that $x \sim y$ for every formula $\bar{D}\{x,y\} \in \Gamma$. We define the *equational closure* $\tilde{\Gamma}$ of a branch Γ as

$$\begin{aligned}\tilde{\Gamma} \;:=\; & \Gamma \cup \{tx \mid \exists x' :\; x' \sim_\Gamma x \wedge tx' \in \Gamma\} \\ & \cup \{rxy \mid \exists x',\, y' :\; x' \sim_\Gamma x \wedge y' \sim_\Gamma y \wedge rx'y' \in \Gamma\}\end{aligned}$$

Clearly, $\tilde{\Gamma}$ is finite if Γ is finite. Reasoning with respect to $\tilde{\Gamma}$ can be implemented efficiently using disjoint-set forests, as demonstrated in [16].

A branch Γ is called *evident* if it satisfies all of the following *evidence conditions*:

$$\begin{aligned}
(t_1 \dot{\wedge} t_2)x \in \Gamma \;&\Rightarrow\; t_1x \in \tilde{\Gamma} \;\wedge\; t_2x \in \tilde{\Gamma} \\
(t_1 \dot{\vee} t_2)x \in \Gamma \;&\Rightarrow\; t_1x \in \tilde{\Gamma} \;\vee\; t_2x \in \tilde{\Gamma} \\
\langle r\rangle_n tx \in \Gamma \;&\Rightarrow\; \exists Y :\; |Y| = n+1 \;\wedge\; DY \in \Gamma \;\wedge\; \{rxy, ty \mid y \in Y\} \subseteq \tilde{\Gamma} \\
[r]_n tx \in \Gamma \;&\Rightarrow\; |\{y \mid rxy \in \tilde{\Gamma},\; ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n \\
E_n tx \in \Gamma \;&\Rightarrow\; \exists Y :\; |Y| = n+1 \;\wedge\; DY \in \Gamma \;\wedge\; \{ty \mid y \in Y\} \subseteq \tilde{\Gamma} \\
A_n tx \in \Gamma \;&\Rightarrow\; |\{y \mid ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \leq n \\
\dot{x}y \in \Gamma \;&\Rightarrow\; x \sim_\Gamma y \\
\dot{\neg}\dot{x}y \in \Gamma \;&\Rightarrow\; x \not\sim_\Gamma y \\
\neg px \in \Gamma \;&\Rightarrow\; px \notin \tilde{\Gamma} \\
\bar{D}X \in \Gamma \;&\Rightarrow\; |X/_{\sim_\Gamma}| < |X| \\
DX \in \Gamma \;&\Rightarrow\; |X/_{\sim_\Gamma}| = |X|
\end{aligned}$$

Note that the evidence condition for $\bar{D}X \in \Gamma$ implies $|X| \geq 2$. A formula s is called *evident on* Γ if Γ satisfies the right-hand side of the evidence condition corresponding to s. For instance, $(t_1 \dot{\wedge} t_2)x$ is evident on Γ if and only if $\{t_1x, t_2x\} \subseteq \tilde{\Gamma}$.

We will now show that evident branches are satisfiable. Given a term t, we write $\mathcal{N}t$ for the set of nominals that occur in t. The notation is extended to sets of terms in the natural way: $\mathcal{N}\Gamma := \bigcup\{\mathcal{N}t \mid t \in \Gamma\}$.

Given a branch Γ, we construct the interpretation $\mathfrak{M}^\Gamma$ by taking as the domain of S the nominals on Γ, and interpreting propositional variables and roles as the smallest sets that are consistent with the respective assertions on Γ. To satisfy the equality constraints on Γ, all nominals that are equivalent modulo $\sim_\Gamma$ are mapped to the same fixed representative.

Let Γ be a branch and let $x_0 \in \mathcal{N}\Gamma$. Let ρ be a function from finite sets of nominals to nominals such that $\rho X \in X$ whenever X is nonempty. We define the interpretation $\mathfrak{M}^\Gamma$ as follows:

$$\begin{aligned}
&\mathfrak{M}^\Gamma \mathrm{S} := \mathcal{N}\Gamma \\
&\mathfrak{M}^\Gamma x := \text{if } x \in \mathcal{N}\Gamma \text{ then } \rho\{y \in \mathcal{N}\Gamma \mid y \sim_\Gamma x\} \text{ else } x_0 \\
&\mathfrak{M}^\Gamma p := \{x \in \mathcal{N}\Gamma \mid px \in \tilde{\Gamma}\} \\
&\mathfrak{M}^\Gamma r := \{(x,y) \in (\mathcal{N}\Gamma)^2 \mid rxy \in \tilde{\Gamma}\}
\end{aligned}$$

Note that in the last two lines of the definition, we interpret the set notation as a convenient description for the respective characteristic functions.

The evidence of $\langle r\rangle_n tx$ (and $E_n tx$) depends on the presence of structurally unrelated and possibly larger formulas DY ($|Y| = n+1$). Similar phenomena will be observed later with our tableau rules (see Fig. 1). Therefore, in the following we will need a measure $\lceil\_\rceil$ on formulas such that, in particular, $\lceil DY\rceil < \lceil\langle r\rangle_n tx\rceil$. Let $\lfloor s\rfloor$ denote the size of a formula s. We define the *order of* s, $\lceil s\rceil$, as follows:

$$\begin{aligned}
\lceil DX\rceil &:= 1 && \\
\lceil \bar{D}X\rceil &:= 1 && \text{if } |X| \leq 2 \\
\lceil \bar{D}X\rceil &:= 2 && \text{if } |X| > 2 \\
\lceil s\rceil &:= 3 + \lfloor s\rfloor && \text{otherwise}
\end{aligned}$$

The case distinction in the definition of $\lceil\bar{D}X\rceil$ is exploited in the proofs of Theorems 3.3 and 4.4.

Theorem 3.1 (Model Existence). *If Γ is evident, then $\mathfrak{M}^\Gamma$ satisfies Γ.*

Proof. For every $s \in \Gamma$, we show that $\mathfrak{M}^\Gamma$ satisfies s by induction on the order of s. □

3.2 Tableau Rules

The tableau rules of our basic calculus $\mathcal{T}$ are defined in Fig. 1. In the rules, we write $\exists x \in X : \Gamma(x)$ for $\Gamma(x_1) \mid \ldots \mid \Gamma(x_n)$, where $X = \{x_1, \ldots, x_n\}$ and $\Gamma(x)$ is a set of formulas parameterized by x. In case $X = \emptyset$, the notation translates to $\bot$. Dually, we write $\forall x \in X : \Gamma(x)$ for $\Gamma(x_1), \ldots, \Gamma(x_n)$ ($X = \{x_1, \ldots, x_n\}$).

The side condition of $\mathcal{R}_\Diamond$ uses the notion of quasi-evidence, which we will introduce in Sect. 3.3. For now, we assume the rule is formulated with the restriction "$\langle r\rangle_n tx$ not evident on Γ".

Note that for $|X| < 2$ the rule $\mathcal{R}_{\bar{D}}$ instantiates to

$$\frac{\bar{D}X}{\bot}$$

A branch Γ is called a *proper extension* of a branch Δ if $\tilde{\Gamma} \supsetneq \tilde{\Delta}$. Note that if Γ is a proper extension of Δ, in particular it holds $\Gamma \supsetneq \Delta$. We implicitly restrict the applicability of the tableau rules such that a rule $\mathcal{R}$ is only applicable to a formula $s \in \Gamma$ if all of the alternative branches $\Delta_1, \ldots, \Delta_n$ resulting from this application are proper extensions of Γ.

Proposition 3.1 (Soundness). *Let $\Delta_1, \ldots, \Delta_n$ be the branches obtained from a branch Γ by a rule of $\mathcal{T}$. Then Γ is satisfiable if and only if there is some $i \in \{1, \ldots, n\}$ such that Δ_i is satisfiable.*

$$\mathcal{R}_{\dot\wedge}\ \frac{(s \mathbin{\dot\wedge} t)x}{sx,\ tx} \qquad\qquad \mathcal{R}_{\dot\vee}\ \frac{(s \mathbin{\dot\vee} t)x}{sx \ \mid\ tx}$$

$$\mathcal{R}_{\Diamond}\ \frac{\langle r\rangle_n tx}{DY,\ \forall y \in Y:\ rxy,\ ty}\ Y \text{ fresh},\ |Y| = n+1,\ \langle r\rangle_n tx \text{ not quasi-evident on } \Gamma$$

$$\mathcal{R}_{\Box}\ \frac{[r]_n tx}{\bar{D}Y \ \mid\ \exists y \in Y:\ ty}\ Y \subseteq \{y \mid rxy \in \tilde{\Gamma}\},\ |Y| = |Y/{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_{E}\ \frac{E_n tx}{DY,\ \forall y \in Y:\ ty}\ Y \text{ fresh},\ |Y| = n+1,\ E_n tx \text{ not evident on } \Gamma$$

$$\mathcal{R}_{A}\ \frac{A_n tx}{\bar{D}Y \ \mid\ \exists y \in Y:\ ty}\ Y \subseteq \mathcal{N}\Gamma,\ |Y| = |Y/{\sim_\Gamma}| = n+1 \qquad \mathcal{R}_{N}\ \frac{\dot{x}y}{\bar{D}\{x,y\}}\ x \neq y$$

$$\mathcal{R}_{\bar{N}}\ \frac{\dot\neg \dot{x}y}{D\{x,y\}}\ x \neq y \qquad \mathcal{R}_{\bar{D}}\ \frac{\bar{D}X}{\exists x,y \in X,\ x \neq y:\ \bar{D}\{x,y\}} \qquad \mathcal{R}_{D}\ \frac{DX}{\bot}\ |X/{\sim_\Gamma}| < |X|$$

$$\mathcal{R}^{\bot}_{\dot\neg}\ \frac{\dot\neg px}{\bot}\ px \in \tilde{\Gamma} \qquad\qquad \mathcal{R}^{\bot}_{\bar{N}}\ \frac{\dot\neg \dot{x}x}{\bot}$$

Γ is the branch to which a rule is applied. "Y fresh" stands for $Y \cap \mathcal{N}\Gamma = \emptyset$.

Fig. 1. Tableau rules for $\mathcal{T}$

3.3 Control

The restrictions on the applicability of the tableau rules given by the evidence conditions are not sufficient for termination. To obtain a terminating calculus, the rule $\mathcal{R}_{\Diamond}$ needs to be restrained further. We do so by weakening the notion of evidence for diamond formulas. The weaker notion, called quasi-evidence, is then used in the side condition of $\mathcal{R}_{\Diamond}$ in place of evidence. Quasi-evidence must be weak enough to guarantee termination of the calculus but strong enough to preserve completeness.

The notions of quasi-evidence used in previous work on pattern-based blocking [15, 14] turn out to be too weak in the presence of graded modalities. For instance, intuitively adapting the notion in [15] would give us the following candidate definition:

A formula $\langle r\rangle_m sx$ is quasi-evident on Γ if there are $y, z_1, \ldots, z_m$ such that $\{ryz_1, sz_1, \ldots, ryz_m, sz_m\} \subseteq \tilde{\Gamma}$ and $\{[r]_n ty \mid [r]_n tx \in \tilde{\Gamma}\} \subseteq \tilde{\Gamma}$. (We also say: $\langle r\rangle_m sx$ is quasi-evident if the corresponding *pattern* $\{\langle r\rangle_m s\} \cup \{[r]_n t \mid [r]_n tx \in \tilde{\Gamma}\}$ is *expanded*).

With this definition of quasi-evidence, no rule of our calculus would apply to the following branch:

$$\Gamma := \{ryz,\ qz,\ [r]_1(p \mathbin{\dot\wedge} \dot\neg p)y,\ \langle r\rangle_0 qx,\ [r]_1(p \mathbin{\dot\wedge} \dot\neg p)x,\ rxu,\ \dot\neg qu\}$$

As Γ is clearly unsatisfiable, the notion of quasi-evidence needs to be adapted.

Given a branch Γ and a role r, an *r-pattern* is a set of expressions of the form μs, where $\mu \in \{\langle r\rangle_n, [r]_n \mid n \in \mathbb{N}\}$. We write $P^r_\Gamma x$ for the largest r-pattern P such that $P \subseteq \{t \mid tx \in \tilde{\Gamma}\}$. We call $P^r_\Gamma x$ the r-pattern of x on Γ. An r-pattern P is *expanded on* Γ if there are nominals x, y such that $rxy \in \tilde{\Gamma}$ and $P \subseteq P^r_\Gamma x$. In this case, we say that the nominal x *expands* P *on* Γ.

A diamond formula $\langle r\rangle_n sx \in \Gamma$ is *quasi-evident on* Γ if it is either evident on Γ or x *has no r-successor on* Γ (i.e., there is no y such that $rxy \in \tilde{\Gamma}$) and $P^r_\Gamma x$ is expanded on Γ. The rule $\mathcal{R}_\Diamond$ can only be applied to diamond formulas that are not quasi-evident.

Note that whenever $\langle r\rangle_n sx \in \Gamma$ is quasi-evident but not evident on Γ, there is a nominal y that expands $P^r_\Gamma x$ on Γ.

We call a branch Γ quasi-evident if it satisfies all of the evidence conditions but the one for diamond formulas, which we replace by:

$$\langle r\rangle_n tx \in \Gamma \;\Rightarrow\; \langle r\rangle_n tx \text{ is quasi-evident on } \Gamma$$

One can show the following lemma:

Lemma 3.1. *Let Γ be a quasi-evident branch and let $\langle r\rangle_n sx \in \Gamma$ be not evident on Γ. Let y be a nominal that expands $P^r_\Gamma x$ on Γ and $\Delta := \Gamma \cup \{rxz \mid ryz \in \tilde{\Gamma}\}$. Then:*

1. *$\forall z :\; rxz \in \tilde{\Delta} \iff ryz \in \tilde{\Gamma}$,*
2. *$\forall m, t :\; \langle r\rangle_m t \in P^r_\Gamma x \implies \langle r\rangle_m tx$ evident on Δ,*
3. *$\langle r\rangle_n sx$ evident on Δ,*
4. *$\forall r', m, t, z :\; \langle r'\rangle_m tz$ evident on $\Gamma \implies \langle r'\rangle_m tz$ evident on Δ,*
5. *Δ quasi-evident.*

Theorem 3.2 (Evidence Completion). *For every quasi-evident branch Γ there is an evident branch Δ such that $\Gamma \subseteq \Delta$.*

Proof. For every branch Γ, we define:

$$\varphi\Gamma := |\{\langle r\rangle_n sx \mid \langle r\rangle_n sx \in \Gamma \wedge \langle t\rangle_n sx \text{ not evident on } \Gamma\}|$$

Let Γ be quasi-evident. We proceed by induction on $\varphi\Gamma$. If $\varphi\Gamma = 0$, then Γ is evident and we are done. Otherwise, there is a diamond $\langle r\rangle_n sx \in \Gamma$ that is not evident on Γ. Let y be a nominal that expands $P^r_\Gamma x$ on Γ, and let $\Gamma' := \Gamma \cup \{rxz \mid ryz \in \tilde{\Gamma}\}$. By Lemma 3.1(3-5), Γ' is quasi-evident and $\varphi\Gamma' < \varphi\Gamma$. So, by the inductive hypothesis, there is some evident branch Δ such that $\Delta \supseteq \Gamma' \supseteq \Gamma$. □

A branch is called *maximal* if it cannot be extended by any tableau rule.

Theorem 3.3 (Quasi-evidence). *Every open and maximal branch in $\mathcal{T}$ is quasi-evident.*

Proof. Let Γ be an open and maximal branch. We show that every $s \in \Gamma$ that is not of the form px or rxy is (quasi-)evident on Γ by induction on the order of s. □

3.4 Termination

We will now show that every tableau derivation is finite. As usual, the main difficulty is bounding the number of applications of generative rules, in particular of $\mathcal{R}_\Diamond$. The present proof is notably more complex than the proofs in [15, 14] since now, an application of $\mathcal{R}_\Diamond$ does not necessarily expand a new pattern. Hence, we need to combine the pattern-counting argument from [15, 14] with a bound on the number of non-expanding applications of $\mathcal{R}_\Diamond$.

Since the rules $\mathcal{R}_{\dot\vee}$, $\mathcal{R}_\Box$, $\mathcal{R}_A$, and $\mathcal{R}_{\bar{D}}$ are all finitely branching, by König's lemma it suffices to show that the construction of every individual branch terminates. Since tableau rule application always produces proper extensions of branches, it then suffices to show that the size (i.e., cardinality) of an individual branch is bounded.

First, we show that the size of a branch Γ is bounded by a function in the number of nominals on Γ. Then, we show that this number itself is bounded from above, completing the termination proof.

We write $\Gamma \overset{\mathcal{R}}{\to} \Delta$ to denote that the branch Δ is obtained from Γ by the rule $\mathcal{R}$. We write $\Gamma \to \Delta$ if Δ is obtained from Γ by a single rule application. We write $\mathcal{S}\Gamma$ for the set of all modal expressions occurring on Γ, possibly as subterms of other expressions, and $\mathrm{Rel}\,\Gamma$ for the set of all roles that occur on Γ.

Crucial for the termination argument is the fact the the tableau rules cannot introduce any modal expressions that do not already occur on the initial branch.

Proposition 3.2. *If Γ, Δ are branches such that Δ is obtained from Γ by any rule of $\mathcal{T}$, then $\mathcal{S}\Delta = \mathcal{S}\Gamma$.*

Let $m_0 = \max\{n \mid \exists r, s, x : \ \langle r\rangle_n sx \in \Gamma \vee [r]_n sx \in \Gamma\}$. For every pair of nominals x, y and every role r, a branch Γ may contain an edge rxy, for every set $X \subseteq \mathcal{N}\Gamma$ where $|X| \leq m_0$, Γ may contain constants DX and $\bar{D}X$ and, for every expression $s \in \mathcal{S}\Gamma$, a formula sx. Hence, the size of Γ is bounded by $|\mathrm{Rel}\,\Gamma| \cdot |\mathcal{N}\Gamma|^2 + 2m_0 \cdot |\mathcal{N}\Gamma|^{m_0} + |\mathcal{S}\Gamma| \cdot |\mathcal{N}\Gamma|$. By Proposition 3.2, we know that $|\mathcal{S}\Gamma|$ and $|\mathrm{Rel}\,\Gamma|$ depend only on the initial branch.

Note that the above bound is exponential in m_0. If, however, we represented distinctness constraints by binary equations and disequations, we could easily give a bound that is independent from m_0 by replacing the summand $2m_0 \cdot |\mathcal{N}\Gamma|^{m_0}$ with $2|\mathcal{N}\Gamma|^2$.

By the above, it suffices to show that $|\mathcal{N}\Gamma|$ is exponentially bounded in the size of the initial formula. We do so by giving a bound on the number of applications of $\mathcal{R}_\Diamond$ and $\mathcal{R}_E$ that can occur in the derivation of a branch, which suffices since $\mathcal{R}_\Diamond$ and $\mathcal{R}_E$ are the only two rules that can introduce new nominals.

We begin by showing that $\mathcal{R}_E$ can be applied at most as many times, as there are distinct modal expressions of the form $E_n s$ on the initial branch. For this purpose, we define a function ψ_E such that $\psi_E\Gamma := \{E_n s \in \mathcal{S}\Gamma \mid \exists x \in \mathcal{N}\Gamma : \ E_n sx \text{ not evident on } \Gamma\}$. Since $|\psi_E\Gamma|$ is bounded from below by 0, it suffices to show that the number decreases with every application of $\mathcal{R}_E$ (and is non-increasing otherwise, which is obvious).

Proposition 3.3. $\Gamma \xrightarrow{\mathcal{R}_E} \Delta \implies |\psi_E \Gamma| > |\psi_E \Delta|$

The proof proceeds analogously to the corresponding arguments in [15, 14].

Now we show that $\mathcal{R}_\Diamond$ can be applied at most finitely often in a derivation. Since there are only finitely many roles, it suffices to show that $\mathcal{R}_\Diamond$ can be applied at most finitely often for each role. Observe that since $\mathcal{R}_\Diamond$ is only applicable to diamond formulas that are not quasi-evident, it holds:

Proposition 3.4. *If $\mathcal{R}_\Diamond$ is applicable to a formula $\langle r\rangle_n sx \in \Gamma$, then either*

1. *x has an r-successor on Γ, or*
2. *$P^r_\Gamma x$ is not expanded on Γ.*

Let Γ and Δ be branches such that Δ is obtained from Γ by applying $\mathcal{R}_\Diamond$ to a formula $\langle r\rangle_n sx \in \Gamma$ such that $P^r_\Gamma x$ is not expanded on Γ. It is easy to see that $P^r_\Delta x$ must be expanded on Δ. Let us call such an application of $\mathcal{R}_\Diamond$ *pattern-expanding.*

Let $\mathrm{Pat}^r\Gamma := \mathcal{P}(\{\langle r\rangle_n s \in \mathcal{S}\Gamma\} \cup \{[r]_n s \in \mathcal{S}\Gamma\})$. In other words, $\mathrm{Pat}^r\Gamma$ contains all the possible sets of r-diamonds and r-boxes from $\mathcal{S}\Gamma$. Since $\Gamma \rightarrow \Delta$ implies $\tilde{\Gamma} \subseteq \tilde{\Delta}$, it holds:

Lemma 3.2. *Let $\Gamma \rightarrow \Delta$ and $P \in \mathrm{Pat}^r\Gamma$. If P is expanded on Γ, then P is expanded on Δ.*

So, for each role r the derivation of a branch has at most $|\mathrm{Pat}^r\Gamma_0|$ pattern-expanding applications of $\mathcal{R}_\Diamond$, where Γ_0 is the initial branch. Clearly, $|\mathrm{Pat}^r\Gamma_0|$ is exponentially bounded in the size of the initial formula.

Hence, it remains to show that a derivation can contain only finitely many applications of $\mathcal{R}_\Diamond$ assuming that none of the applications is pattern-expanding. We say a nominal x has a *successor* on Γ if x has an r-successor on Γ for any role r. A set of nominals X has a successor on Γ if there is some $x \in X$ that has a successor on Γ. We define

$$\psi^X_\Diamond \Gamma := |\{\langle r\rangle_n s \in \mathcal{S}\Gamma \mid \exists x \in X:\ \langle r\rangle_n sx \text{ not evident on } \Gamma\}|$$

and

$$\psi_\Diamond \Gamma := \sum_{\substack{X \in \mathcal{N}\Gamma/\sim_\Gamma \\ X \text{ has a successor on } \Gamma}} \psi^X_\Diamond \Gamma \ .$$

Proposition 3.5. *Let $\Gamma \rightarrow \Delta$ such that Δ is obtained from Γ by some rule application other than a pattern-expanding application of $\mathcal{R}_\Diamond$.*

1. *If Δ is obtained from Γ by $\mathcal{R}_\Diamond$, then $\psi_\Diamond\Gamma > \psi_\Diamond\Delta$.*
2. *Otherwise, $\psi_\Diamond\Gamma \geq \psi_\Diamond\Delta$.*

This completes the termination proof. Since the cardinalities of the sets $\mathrm{Pat}^r\Gamma$ are exponentially bounded in the size n_0 of the initial formula, $|\psi_E\Gamma|$ is polynomial in n_0, and $\psi_\Diamond\Gamma$ polynomial in $|\Gamma|$ and n_0, $|\mathcal{N}\Gamma|$ is exponentially bounded in n_0. Since $|\Gamma|$ is polynomial in $|\mathcal{N}\Gamma|$, we conclude that $|\Gamma|$ is at most exponential in n_0. By cumulativity, the construction of Γ terminates in at most exponentially many steps in n_0. This suffices to give us a NExpTime complexity bound for the decision procedure based on the calculus.

4 Adding Reflexivity, Transitivity and Role Inclusion

We now extend $\mathcal{T}$ to deal with reflexivity, transitivity and inclusion assertions. As in related work on description logic [5, 8, 10, 9], we restrict our modal expressions to contain no graded boxes for roles that have transitive subroles.

We define $\subseteq^*_\Gamma$ as the smallest reflexive and transitive relation such that $r \subseteq^*_\Gamma r'$ whenever $r \sqsubseteq r' \in \Gamma$. A role r is called *simple* if there is no r' such that $r' \subseteq^*_\Gamma r$ and $Tr' \in \Gamma$. Observe that all subroles of a simple role are in turn simple.

Our branches may now contain inclusion, reflexivity and transitivity assertions:

$$s \ ::= \ tx \mid rxy \mid DX \mid \bar{D}X \mid \bot \mid r \sqsubseteq r' \mid Rr \mid Tr$$

The modal expressions t in formulas of the form tx are restricted to contain no boxes $[r]_n s$ with $n > 0$ unless r is simple.

Following the ideas in [5, 8, 10, 9], we introduce the *induced transition relation* $\trianglerighteq^r_\Gamma$ to reason about accessibility in the presence of inclusion axioms. Intuitively, $x \trianglerighteq^r_\Gamma y$ means that in every model of Γ, y is accessible from x via r.

4.1 Extending Evidence

To account for the new types of formulas, we extend the evidence conditions as follows:

$$\begin{aligned}
r \sqsubseteq r' \in \Gamma \ &\Rightarrow\ \forall x, y \in \mathcal{N}\Gamma :\ rxy \in \tilde{\Gamma} \Rightarrow r'xy \in \tilde{\Gamma} \\
Rr \in \Gamma \ &\Rightarrow\ \forall x \in \mathcal{N}\Gamma :\ rxx \in \tilde{\Gamma} \\
Tr \in \Gamma \ &\Rightarrow\ \forall x, y, z \in \mathcal{N}\Gamma :\ rxy \in \tilde{\Gamma} \wedge ryz \in \tilde{\Gamma} \Rightarrow rxz \in \tilde{\Gamma}
\end{aligned}$$

It is easy to see that if Γ satisfies the extended evidence conditions, $\mathfrak{M}^\Gamma$ will satisfy the new formulas. Hence, Theorem 3.1 adapts to the extended system.

Theorem 4.1 (Model Existence). *If Γ is evident, then $\mathfrak{M}^\Gamma$ satisfies Γ.*

4.2 Pre-evidence

To account for the new evidence conditions, one could imagine the following rules.

$$\frac{r \sqsubseteq r',\ rxy}{r'xy} \qquad \frac{Rr}{rxx}\ x \in \mathcal{N}\Gamma \qquad \frac{Tr,\ rxy,\ ryz}{rxz}$$

In the presence of blocking, however, the rules are problematic. In particular, the rule for reflexivity renders the notion of quasi-evidence that we use for $\mathcal{T}$ ineffective to ensure termination. Once we add a reflexive edge rxx to a branch Γ, x will have an r-successor on Γ, meaning quasi-evidence will coincide with evidence for all r-diamonds on x. Similarly, the rule for transitivity is known to be incomplete in the presence of blocking [14].

We solve the problem by defining a weaker notion of evidence, called *pre-evidence*. To satisfy the pre-evidence conditions, we do not have to explicitly add reflexive or transitive edges during tableau construction. We will extend our tableau rules and the notion of quasi-evidence such that every open and maximal branch in the extended calculus can be completed to a pre-evident branch, which in turn can be made evident by adding the implicit edges.

We define the relation $\rhd^r_\Gamma$ as the least relation such that:

$$\begin{aligned} rxy \in \tilde{\Gamma} \quad &\Rightarrow \quad x \rhd^r_\Gamma y \\ r' \sqsubseteq r \in \Gamma,\ x \rhd^{r'}_\Gamma y \quad &\Rightarrow \quad x \rhd^r_\Gamma y \end{aligned}$$

The relation $\rhd^r_\Gamma$ does not account for reflexivity. To do so, we extend it as follows:

$$\unrhd^r_\Gamma \ := \ \begin{cases} \rhd^r_\Gamma \cup \{(x,y) \mid x,y \in \mathcal{N}\Gamma \wedge x \sim_\Gamma y\} & \text{if } \exists r' :\ r' \subseteq^*_\Gamma r \wedge Rr' \in \Gamma \\ \rhd^r_\Gamma & \text{otherwise} \end{cases}$$

The *pre-evidence conditions* are obtained from the evidence conditions by omitting the conditions for inclusion and reflexivity assertions and replacing the conditions for diamonds, boxes and transitivity assertions as follows:

$$\begin{aligned} \langle r\rangle_n tx \in \Gamma \ &\Rightarrow\ \exists Y\colon\ |Y| = n+1 \ \wedge\ DY \in \tilde{\Gamma} \ \wedge\ \forall y \in Y:\ x \unrhd^r_\Gamma y \wedge ty \in \tilde{\Gamma} \\ [r]_n tx \in \Gamma \ &\Rightarrow\ |\{y \mid x \unrhd^r_\Gamma y,\ ty \notin \tilde{\Gamma}\}/_{\sim_\Gamma}| \le n \\ Tr \in \Gamma \ &\Rightarrow\ \forall x,y:\ [r']_0 tx \in \tilde{\Gamma} \wedge r \subseteq^*_\Gamma r' \wedge x \rhd^r_\Gamma y \Rightarrow [r]_0 ty \in \tilde{\Gamma} \end{aligned}$$

Note that we do not need pre-evidence conditions for inclusion or reflexivity assertions as their semantics is taken care of by the way we define the relation $x \unrhd^r_\Gamma y$. Pre-evidence of individual formulas is defined analogously to the corresponding evidence notion.

We now show that every pre-evident branch can be extended to an evident branch. Let the *evidence closure* $\hat{\Gamma}$ of a branch Γ be defined as the least superset of Γ such that:

$$\begin{aligned} x \unrhd^r_\Gamma y \ &\Rightarrow\ rxy \in \hat{\Gamma} \\ Tr \in \hat{\Gamma} \wedge rxy \in \hat{\Gamma} \wedge ryz \in \hat{\Gamma} \ &\Rightarrow\ rxz \in \hat{\Gamma} \\ r \sqsubseteq r' \in \hat{\Gamma} \wedge rxy \in \hat{\Gamma} \ &\Rightarrow\ r'xy \in \hat{\Gamma} \end{aligned}$$

Note that by construction, we have $rxy \in \hat{\hat{\Gamma}} \iff rxy \in \hat{\Gamma}$.

Lemma 4.1. *Let Γ be a branch and r be simple on Γ. Then $x \unrhd^r_\Gamma y \iff rxy \in \hat{\Gamma}$*

Lemma 4.2. *Let Γ be a branch and let $rxy \in \hat{\Gamma}$. Then either $x \unrhd^r_\Gamma y$, or there is an r' such that $\{r' \sqsubseteq r,\ Tr'\} \subseteq \Gamma$ and*

$$\exists\, n{\ge}2\ \exists x_1,\ldots,x_n\colon\ x_1 = x \ \wedge\ x_n = y \ \wedge\ \forall 1{\le}i{<}n:\ x_i \rhd^{r'}_\Gamma x_{i+1} \ .$$

Theorem 4.2 (Evidence Completion). *Γ pre-evident $\Longrightarrow$ $\hat{\Gamma}$ evident*

Proof. Straightforward, using Lemmas 4.1 and 4.2. □

$$\mathcal{R}_{\Box}\ \frac{[r]_n tx}{\bar{D}Y \ \mid\ \exists y \in Y:\ ty}\ Y \subseteq \{y \mid x \trianglerighteq^r_\Gamma y\},\ |Y| = |Y/_{\sim_\Gamma}| = n+1$$

$$\mathcal{R}_T\ \frac{Tr,\ [r']_0 tx}{[r]_0 ty}\ r \sqsubseteq^*_\Gamma r',\ x \rhd^r_\Gamma y$$

Fig. 2. New rules for $\mathcal{T}_{\sqsubseteq}$

4.3 Tableau Rules

The tableau rules for the extended calculus $\mathcal{T}_{\sqsubseteq}$ in Fig. 2 replace the original rule $\mathcal{R}_{\Box}$ from Fig. 1 and add a new rule $\mathcal{R}_T$, which is necessary to achieve the pre-evidence condition for transitivity assertions. While the formulation of $\mathcal{R}_{\Diamond}$ remains unchanged, the rule will now have to use an adapted notion of quasi-evidence, which will be introduced in Sect. 4.4. For now, we assume $\mathcal{R}_{\Diamond}$ is formulated with the restriction "$\langle r\rangle_n tx$ not pre-evident on Γ" instead. Again, it is not hard to verify that the extended rules are sound.

4.4 Control

As it turns out, in the presence of role inclusion we have to modify the definition of patterns. It no longer suffices to consider patterns separately for each role. This is due to the fact that now, different roles may be constrained by inclusion assertions. Consider, for instance, the unsatisfiable branch

$$\Gamma := \{r \sqsubseteq r',\ \langle r\rangle_0 px,\ \langle r'\rangle_0 \dot{\neg} px,\ [r']_1(p \dot{\wedge} \dot{\neg} p)x,\ r'xy,\ \dot{\neg} py,\ \langle r\rangle_0 pz,\ rzu,\ pu\}$$

According to our previous notion of quasi-evidence, $\langle r\rangle_0 px$ is quasi-evident on Γ as x has no r-successor (even if we extend the set of successors to $\{y \mid x \rhd^r_\Gamma y\}$) and $P^r_\Gamma x$ is expanded. Since the other two diamonds on Γ are evident, Γ is quasi-evident, witnessing the incompleteness of our previous definition of patterns.

Hence, we redefine the notion of a pattern as follows. Given a branch Γ, a *pattern* is a set of terms of the form μs, where $\mu \in \{\langle r\rangle_n, [r]_n \mid r \in \mathrm{Rel}\,\Gamma,\ n \in \mathbb{N}\}$. We write $P_\Gamma x$ for the largest pattern P such that $P \subseteq \{t \mid tx \in \tilde{\Gamma}\}$. We call $P_\Gamma x$ the pattern of x on Γ. A pattern P is *expanded on* Γ if there are nominals x, y and a role r such that $x \rhd^r_\Gamma y$ and $P \subseteq P_\Gamma x$. In this case, we say that x *expands* P *on* Γ. Note that here we use the relation $\rhd^r_\Gamma$ rather than $\trianglerighteq^r_\Gamma$. Otherwise, we would get the same problems with termination as outlined in Sect. 4.2.

A diamond formula $\langle r\rangle_n sx$ is *quasi-evident on* Γ if it is either pre-evident on Γ or x has no *successor* on Γ (i.e., there is no y such that for any r, $x \rhd^r_\Gamma y$) and $P_\Gamma x$ is expanded on Γ. As before, we restrain the rule $\mathcal{R}_{\Diamond}$ such that it can only be applied to diamond formulas that are not quasi-evident, and call a branch Γ quasi-evident if it satisfies all of the pre-evidence conditions but the one for diamond formulas, which we again replace by

$$\langle r\rangle_n tx \in \Gamma \ \Rightarrow\ \langle r\rangle_n tx \text{ is quasi-evident on } \Gamma$$

but now with the adapted notion of quasi-evidence.

Lemma 4.3. *Let x, y, u, v be nominals and Γ, Δ branches such that $\{r \mid rxy \in \tilde{\Gamma}\} = \{r \mid ruv \in \tilde{\Delta}\}$. Then, for every r, $x \rhd_\Gamma^r y \;\Leftrightarrow\; u \rhd_\Delta^r v$.*

Lemma 4.4. *Let Γ be a quasi-evident branch and let $\langle r\rangle_n sx$ be not pre-evident on Γ. Let y expand $P_\Gamma x$ on Γ and $\Delta := \Gamma \cup \{r'xz \mid r'yz \in \tilde{\Gamma}\}$. Then:*

1. $\forall r', z:\; x \rhd_\Delta^{r'} z \iff y \rhd_\Gamma^{r'} z$ *and* $x \trianglerighteq_\Delta^{r'} z \iff y \trianglerighteq_\Gamma^{r'} z$,
2. $\forall r', m, t:\; \langle r'\rangle_m t \in P_\Gamma x \implies \langle r'\rangle_m tx$ *pre-evident on* Δ,
3. $\langle r\rangle_n sx$ *pre-evident on* Δ,
4. $\forall r', m, t, z:\; \langle r'\rangle_m tz$ *pre-evident on* $\Gamma \implies \langle r'\rangle_m tz$ *pre-evident on* Δ,
5. Δ *quasi-evident.*

Proof. Analogous to the proof of Lemma 3.1, Lemma 4.3 being used for (1). □

Theorem 4.3 (Pre-evidence Completion). *For every quasi-evident branch Γ there is a pre-evident branch Δ such that $\Gamma \subseteq \Delta$.*

Proof. Proceeds analogously to the proof of Theorem 3.2 with Lemma 4.4 in place of Lemma 3.1. □

Theorem 4.4 (Quasi-evidence). *Every open and maximal branch in $\mathcal{T}_\sqsubseteq$ is quasi-evident.*

Proof. Proceeds analogously to the proof of Theorem 3.3. □

While requiring some adaptations, the termination proof for $\mathcal{T}_\sqsubseteq$ is mostly analogous to the proof for $\mathcal{T}$.

5 Conclusion

We have presented a terminating tableau calculus for graded hybrid logic with global modalities and role hierarchies. Following [19, 20, 14], our calculus is cumulative, representing state equality abstractly via an equivalence relation (declarative approach). The existing calculi for equivalent and stronger logics [8, 10, 9] work on possibly cyclic graph structures and treat equality by destructive graph transformation during tableau construction (procedural approach). The procedural approach encompasses algorithmic decisions that are not present in the more abstract declarative approach. From a declarative calculus we can always obtain a procedural system by refinement.

Exploiting an extended pattern-based blocking technique and the cumulativity of our calculus, we have proved a NExpTime complexity bound for the associated decision procedure. To ensure termination of pattern-based blocking in the presence of reflexivity, we differentiated between the induced transition relation $\trianglerighteq_\Gamma^r$ and its non-reflexive counterpart $\rhd_\Gamma^r$. The implementation of pattern-based blocking for a hybrid language with global modalities [16] reveals its considerable practical potential. We consider it a promising project to implement the extended version of pattern-based blocking presented in this paper and compare its performance to that of established blocking techniques.

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