Transfinite Constructions in Classical Type Theory

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We study a transfinite construction we call tower construction in classical type theory. The construction is inductive and applies to partially ordered types. It yields the set of all points reachable from a starting point with an increasing successor function and a family of admissible suprema. Based on the construction, we obtain type-theoretic versions of the theorems of Zermelo (well-orderings), Hausdorff (maximal chains), and Bourbaki and Witt (fixed points). The development is formalized in Coq assuming excluded middle.

1 Introduction

We are interested in type-theoretic versions of set-theoretic theorems involving transfinite constructions. Here are three prominent examples we will consider in this paper:

- **Zermelo 1904** [12, 13] Every set with a choice function can be well-ordered.
- **Hausdorff 1914** [5] Every poset with a choice function has a maximal chain.
- **Bourbaki-Witt 1951** [2, 11, 9] Every increasing function on a chain-complete poset has a fixed point.

All three results can be obtained in type theory with a transfinite construction we call tower construction. We start with a partial order $\le$ on a type X and a *successor function* $f : X \to X$ such that $x \le fx$ for all x. We also assume a subset-closed family of *admissible sets* (i.e., unary predicates on X) and a **join function** $\sqcup$ that yields the suprema of admissible sets. Now for each point a in X we inductively define a set Σ_a we call the tower for a:

$$\frac{}{a \in \Sigma_a} \qquad \frac{x \in \Sigma_a}{fx \in \Sigma_a} \qquad \frac{p \subseteq \Sigma_a \qquad p \text{ admissible and inhabited}}{\sqcup p \in \Sigma_a}$$

For every tower we show the following:

1. Σ_a is well-ordered by $\leq$.
2. If Σ_a is admissible, then $\sqcup\Sigma_a$ is the greatest element of Σ_a.
3. $x \in \Sigma_a$ is a fixed point of f iff x is the greatest element of Σ_a.

The proofs make frequent use of excluded middle but no other assumptions are needed.

The tower construction formalizes Cantor's idea of transfinite iteration in type theory. No notion of ordinals is used. In axiomatic set theory, an instance of the tower construction appears first in 1908 in dualized form in Zermelo's second proof [13] of the well-ordering theorem. The general form of the tower construction in set theory was identified by Bourbaki [2] in 1949. Bourbaki [2] defines Σ_a as the intersection of all sets closed under the rules of our inductive definition and proves the three results stated above. Felscher [3] discusses the tower construction in set theory and gives many historical references. Felscher reports that the tower construction was already studied in 1909 by Hessenberg [6] in almost general form (set inclusion as order and union as supremum).

The Bourbaki-Witt theorem stated at the beginning of this section does not appear in Bourbaki [2]. The theorem is, however, an immediate consequence of the results Bourbaki [2] shows for towers (see above). As admissible sets we take chains; Result 1 tells us that Σ_a is a chain; hence it follows with Results 2 and 3 that $\sqcup\Sigma$ is a fixed point of f. In fact, the argument gives us a generalized version of the Bourbaki-Witt fixed point theorem that requires suprema only for well-ordered subsets and that asserts the existence of a fixed point above any given point. Explicit statements of the Bourbaki-Witt fixpoint theorem appear in Witt [11] and Lang [9]. In contrast to Bourbaki [2], who like us sees the tower construction as the main object of interest, Witt and Lang see the fixed point theorem as the main result and hide the tower construction in the proof of the theorem.

The theorems of Zermelo and Hausdorff mentioned above can also be obtained with the tower construction (Bourbaki [2] argues the case for Zermelo's theorem and Zorn's lemma, a better known variant of Hausdorff's theorem). For both theorems, we start with a base type B and take for X the type of all sets over B. We assume that the sets over B are extensional. As ordering we take set inclusion and as admissible sets we take all families of sets over B. For Zermelo's theorem, the successor function adds an element as determined by the given choice function. The well-ordering of the tower $\Sigma_\emptyset$ for the empty set then induces a well-ordering of B. For Hausdorff's theorem, we define the successor function as the function that based on the given choice function adds an element such that a chain is obtained if this is possible. Then $\sqcup\Sigma_{\emptyset}$ is a maximal chain.

The present paper is organized as follows. We first study a specialized tower construction that suffices for the proofs of Zermelo's and Hausdorff's theorem. We present the specialized tower construction in addition to the general construction since it comes with simpler proofs. The specialized tower construction operates on sets and uses a successor function that adds at most one element. We obtain proofs of Hausdorff's and Zermelo's theorem. From Zermelo's theorem we obtain a result asserting the existence of well-ordered extensions of well-founded relations. We then restart in a more abstract setting and study the general tower construction. We prove that towers are well-ordered and obtain the Bourbaki-Witt fixed point theorem.

The development of this paper is formalized in Coq assuming excluded middle and can be found at `https://www.ps.uni-saarland.de/extras/itp15`. The development profits much from Coq's support for inductive definitions. The material in this paper is a perfect candidate for formalization since it is abstract and pretty formal anyway. The interactive support Coq provides for the construction and verification of the often technical proofs turned out to be beneficial. The fact that Coq comes without built-in assumptions like choice and extensionality also proved to be beneficial since this way we could easily identify the minimal assumptions needed for the development.

We mention some related work in type theory. A type-theoretic proof of Zermelo's theorem based on Zorn's lemma appears in Isabelle's standard library [8] (including well-ordered extensions of well-founded relations). Ilik [7] presents a type-theoretic proof of Zermelo's theorem formalized in AgdaLight following Zermelo's 1904 proof. Bauer and Lumsdaine [1] study the Bourbaki-Witt fixed point principle in an intuitionistic setting.

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2 Sets as Unary Predicates

Assumption 1 We assume excluded middle throughout the paper.

Let X be a type. A **set over X** is a unary predicate on X. We write

$$\mathit{set}\,X \;:=\; X \to \mathit{Prop}$$

for the type of sets over X and use familiar notations for sets:

$$\begin{aligned} x \in p &:= px & \{x \mid s\} &:= \lambda x.s \\ p \subseteq q &:= \forall x \in p.\, x \in q & \{x\} &:= \lambda z.\, z = x \\ p \subset q &:= p \subseteq q \wedge \exists x \in q.\, x \notin p & X &:= \lambda x.\top \qquad \emptyset := \lambda x.\bot \\ p \cap q &:= \{x \mid x \in p \wedge x \in q\} & p \cup q &:= \{x \mid x \in p \vee x \in q\} \\ p \setminus q &:= \{x \mid x \in p \wedge x \notin q\} & \neg p &:= \{x \mid x \notin p\} \end{aligned}$$

We call a set **inhabited** if it has at least one member, and **unique** if it has at most one member. A **singleton** is a set that has exactly one member. We call a set **empty** if it has no member. We call two sets p and q **comparable** if either $p \subseteq q$ or $q \subseteq p$.

We call a type X **extensional** if two sets over X are equal whenever they have the same elements. If we assume excluded middle, sets over an extensional type are very much like the familiar mathematical sets.

Fact 2 Let p and q be sets over an extensional type X. Then:

1. If $p \subseteq q$ and $q \subseteq p$, then $p = q$.
2. $p \subseteq q$ iff $p \subset q$ or $p = q$.
3. $p \subset q$ iff $p \subseteq q$ and $p \neq q$.
4. $\neg\neg p = p$, $\neg(p \cap q) = \neg p \cup \neg q$, $\neg(p \cup q) = \neg p \cap \neg q$, and $p \setminus q = p \cap \neg q$.

By a **family over X** we mean a set over *set* X. We define **intersection** and **union** of families as one would expect:

$$\bigcap F := \{x \mid \forall p \in F.\, x \in p\} \qquad \bigcup F := \{x \mid \exists p \in F.\, x \in p\}$$

Note that the members of a family are ordered by inclusion.

3 Orderings and Choice Functions

Let X be a type.

Let R be a binary predicate on X and p be a set over X. We define

$$LR\, p := \{x \in p \mid \forall y \in p.\, Rxy\}$$

and speak of the **set of least elements for R and p**.

Let p be a set over X. A **partial ordering of p** is a binary predicate $\leq$ on X satisfying the following properties for all $x, y, z \in p$:

- *Reflexivity:* $x \leq x$.
- *Antisymmetry:* If $x \leq y$ and $y \leq x$, then $x = y$.
- *Transitivity:* If $x \leq y$ and $y \leq z$, then $x \leq z$.

A **linear ordering of p** is a partial ordering $\leq$ of p such that for all $x, y \in p$ either $x \leq y$ or $y \leq x$. A **well-ordering of p** is a partial ordering $\leq$ of p such that every inhabited subset of p has a least element (i.e., $L(\leq)q$ is inhabited whenever $q \subseteq p$ is inhabited). Note that every well-ordering of p is a linear ordering of p.

A **partial ordering of X** is a partial ordering of $\lambda x.\top$. **Linear orderings of X** and **well-orderings of X** are defined analogously.

We define the notation $x < y := x \leq y \wedge x \neq y$.

Fact 3 (Inclusion and Families) Let F be a family over an extensional type X. Then inclusion $\lambda pq.\, p \subseteq q$ is a partial ordering of F. Moreover:

1. Inclusion is a linear ordering of F iff two members of F are always comparable.
2. Inclusion is a well-ordering of F iff $\bigcap G \in G$ for every inhabited $G \subseteq F$.

A **choice function for X** is a function $\gamma : \mathit{set}\, X \to \mathit{set}\, X$ such that $\gamma p \subseteq p$ for every p and γp is a singleton whenever p is inhabited. Our definition of a choice function is such that no description operator is needed to obtain a choice function from a well-ordering.

Fact 4 If $\leq$ is a well-ordering of X, then $L(\leq)$ is a choice function for X.

4 Special Towers

An **extension function** for a type X is a function $\eta : \mathit{set}\, X \to \mathit{set}\, X$ such that $\eta p \subseteq \neg p$ for all p. An extension function η is called **unique** if ηp is unique for every p.

Assumption 5 Let X be an extensional type and η be a unique extension function for X.

We define $p^+ := p \cup \eta p$ and speak of **adjunction**. We inductively define a family Σ over X:

$$\frac{F \subseteq \Sigma}{\bigcup F \in \Sigma} \qquad\qquad \frac{p \in \Sigma}{p^+ \in \Sigma}$$

We refer to the definition of Σ as **specialized tower construction** and call the family Σ **tower** and the elements of Σ **stages**. We have $\bigcup \emptyset = \emptyset \in \Sigma$. Note that $\emptyset$ is the least stage and $\bigcup \Sigma$ is the greatest stage. Also note that Σ is a minimal family over X that is closed under union and adjunction.

Fact 6 $\eta(\bigcup \Sigma) = \emptyset$.

Proof Suppose $x \in \eta(\bigcup \Sigma)$. Since $(\bigcup \Sigma)^+ \subseteq \bigcup \Sigma$, we have $x \in \bigcup \Sigma$. Contradiction since η is an extension function. ∎

We show that Σ is linearly ordered by inclusion. We base the proof on two lemmas.

Lemma 7 Let a set p and a family F be given such that p is comparable with every member of F. Then p is comparable with $\bigcup F$.

Proof Suppose $\bigcup F \nsubseteq p$. Then there exists $q \in F$ such that $q \nsubseteq p$. Thus $p \subseteq q$ by assumption. Hence $p \subseteq \bigcup F$. ■

Lemma 8 Let $p \subseteq q^+$ and $q \subseteq p^+$. Then p^+ and $q^+$are comparable.

Proof The claim is obvious if $\eta p \subseteq q$ or $\eta q \subseteq p$. Otherwise, we have $x \in \eta p \setminus q$ and $y \in \eta q \setminus p$. We show $p = q$, which yields the claim.

Let $z \in p$. Then $z \in q^+$. Suppose $z \in \eta q$. Then $z = y$ by uniqueness of ηq. Contradiction since $y \notin p$.

The other direction is analogous. ■

Fact 9 (Linearity) Σ is linearly ordered by inclusion.

Proof Let p and q be in Σ. We show by nested induction on $p \in \Sigma$ and $q \in \Sigma$ that p and q are comparable. There are four cases. The cases where p or q is a union follow with Lemma 7. The remaining case where $p = p_1^+$ and $q = q_1^+$ follows with Lemma 8 by exploiting the inductive hypotheses for the pair p_1 and q and the pair q_1 and p. ■

Fact 10 (Inclusion) Let $p, q \in \Sigma$ and $p \subset q$. Then $p^+ \subseteq q$.

Proof By Linearity we have $p^+ \subseteq q$ or $q \subseteq p^+$. The first case is trivial. For the second case we have $p \subset q \subseteq p^+ = p \cup \eta p$. Since ηp is unique, we have $q = p^+$. The claim follows. ■

Fact 11 (Greatest Element)
Let $p^+ = p \in \Sigma$. Then p is the greatest element of Σ.

Proof Let $q \in \Sigma$. We show $q \subseteq p$ by induction on $q \in \Sigma$.

Union. Let $F \subseteq \Sigma$. We show $\bigcup F \subseteq p$. Let $r \in F$. We show $r \subseteq p$. The claim follows by the inductive hypothesis.

Adjunction. Let $q \in \Sigma$. We show $q^+ \subseteq p$. We have $q \subseteq p$ by the inductive hypothesis. If $q = p$, $q^+ = p^+ = p$ by the assumption. If $q \neq p$, we have $q^+ \subseteq p$ by Fact 10. ■

Fact 12 Let $p^+ = p \in \Sigma$. Then $p = \bigcup \Sigma$.

Proof We know that $\bigcup\Sigma$ is the greatest element of Σ. Since greatest elements are unique, we know $p = \bigcup\Sigma$ by Fact 11. ∎

Lemma 13 Let $F \subseteq \Sigma$ be inhabited. Then $\bigcap F \in F$.

Proof By contradiction. Suppose $\bigcap F \notin F$. Then $\bigcap F \subset q$ whenever $q \in F$. We define

$$p \;:=\; \bigcup\{\, q \in \Sigma \mid q \subseteq \bigcap F \,\}$$

We have $p \subseteq \bigcap F$. Since $\bigcap F \notin F$, we have $p \subset q$ whenever $q \in F$. Thus $p^+ \subseteq \bigcap F$ by Fact 10. Hence $p^+ \subseteq p$ by the definition of p. Thus $p^+ = p$. Let $q \in F$. We have $p \subseteq \bigcap F \subseteq q \subseteq \bigcup\Sigma = p$ by Fact 12. Thus $\bigcap\Sigma = q \in F$. Contradiction. ∎

Theorem 14 (Well-Ordering) Σ is well-ordered by inclusion.

Proof Follows with Lemma 13 and Fact 3. ∎

Fact 15 (Intersection) Σ is closed under intersections.

Proof Follows with Lemma 13 and Fact 16. ∎

We call an extension function η **exhaustive** if ηp is inhabited whenever $\neg p$ is inhabited.

Fact 16 If η is exhaustive, then $\bigcup\Sigma = X \in \Sigma$.

Proof Follows with Facts 6. ∎

5 Hausdorff's Theorem

Assumption 17 Let R be a binary predicate on X and γ be a choice function for X.

We call a set p over X a **chain** if for all $x, y \in p$ either Rxy or Ryx. We call a family over X **linear** if it is linearly ordered by inclusion.

Fact 18 The union of a linear family of chains is a chain.

We show that the tower construction gives us a maximal chain. We choose the extension function $\eta p := \gamma(\lambda x.\, x \notin p \wedge \mathit{chain}\ (p \cup \{x\}))$. Clearly, η is unique. Moreover, all stages are chains since the rules of the tower construction yield chains when applied to chains. That the union rule yields a chain follows from Fact 18 and the linearity of Σ. We have $\eta(\bigcup\Sigma) = \emptyset$ for the greatest stage. By the definition of η it now follows that $\bigcup\Sigma$ is a maximal chain. This completes our proof of Hausdorff's theorem.

Theorem 19 (Existence of Maximal Chains) Let X be an extensional type with a choice function. Then every binary predicate on X has a maximal chain.

Proof Follows from the development above. ∎

Hausdorff's theorem can be strengthened so that one obtains a maximal chain extending a given chain. For the proof one uses an extension function that gives preference to the elements of the given chain. If q is the given chain, the extension function $\lambda p.\, \gamma(\lambda x.\, x \notin p \land \mathit{chain}\,(p \cup \{x\}) \land (x \in q \lor q \subseteq p))$ does the job.

We remark that Zorn's lemma is a straightforward consequence of Hausdorff's theorem.

6 Zermelo's Theorem

We can obtain a well-ordering of $\bigcup\Sigma$ by constructing an injective embedding of $\bigcup\Sigma$ into Σ. This gives us a well-ordering of X if the extension function is exhaustive. Since a choice function γ for X gives us a unique and exhaustive extension function $\lambda p.\, \gamma(\neg p)$, we have arrived at a proof of Zermelo's theorem.

We define the **stage for x** as the greatest stage not containing x:

$$\overline{x} \;:=\; \bigcup\{\, q \in \Sigma \mid x \notin q \,\}$$

Fact 20 Let $x \in \bigcup\Sigma$. Then $\eta\overline{x} = \{x\}$.

Proof Since η is unique it suffices to show $x \in \eta\overline{x}$. Suppose $x \notin \eta\overline{x}$. Then $\overline{x}^+ \subseteq \overline{x}$ since $\overline{x}^+$ is a stage not containing x. Thus $\overline{x} = \bigcup\Sigma$ by Fact 12. Contradiction since $x \notin \overline{x}$. ∎

Fact 21 (Injectivity) Let $x, y \in \bigcup\Sigma$ and $\overline{x} = \overline{y}$. Then $x = y$.

Proof Follows with Fact 20 and the uniqueness of η since $\eta\overline{x} = \eta\overline{y}$. ∎

Theorem 22 $x \leq_\eta y := \overline{x} \subseteq \overline{y}$ is a well-ordering of $\bigcup\Sigma$.

Proof Follows with Fact 21 and Theorem 14. ∎

We prove some properties of the well-ordering we have obtained for $\bigcup\Sigma$.

Fact 23 Let $x \in p \in \Sigma$. Then $\overline{x} \subset p$.

Proof By Linearity it suffices to show that $p \subseteq \overline{x}$ is contradictory, which is the case since $x \in p$ and $x \notin \overline{x}$. ∎

Fact 24 Let $x \in \bigcup\Sigma$. Then $\overline{x} = \{ z \in \bigcup\Sigma \mid \overline{z} \subset \overline{x} \}$.

Proof Let $z \in \overline{x}$. We have $\overline{z} \subset \overline{x}$ by Fact 23. Let $z \in \bigcup\Sigma$ and $\overline{z} \subset \overline{x}$. We show $z \in \overline{x}$. We have $z \in \eta\overline{z} \subseteq \overline{x}$ by Facts 20 and 10. ∎

Let $\leq$ be a partial ordering of X. We define the **segment for x and $\leq$** as $S_x := \{ z \mid z < x \}$. We say that a well-ordering $\leq$ **agrees** with a choice function γ if $\gamma(\neg S_x) = \{x\}$ for every x.

Theorem 25 (Existence of Well-Orderings) Let X be an extensional type and γ be a choice function for X. Then there exists a well-ordering of X that agrees with γ.

Proof Let $\eta p := \gamma(\neg x)$. Then η is a unique and exhaustive extension function for X. We have $X = \bigcup\Sigma$ by Fact 16. Thus $\leq_\eta$ is a well-ordering of X by Theorem 22. Let S_x^η be the segment for x and $\leq_\eta$. We have $S_x^\eta = \overline{x}$ by Fact 24. Moreover, $\leq_\eta$ agrees with γ since $\gamma(\neg S_x^\eta) = \eta(S_x^\eta) = \eta\,\overline{x} = \{x\}$ by Facts 24 and 20. ∎

One can show that there exists at most one well-ordering of X that agrees with a given choice function. Thus our construction yields the unique well-ordering that agrees with the given choice function. This is also true for the well-orderings obtained with Zermelo's proofs [12, 13].

We show that with Theorem 25 we can get well-ordered extensions of well-founded predicates. Let R be a binary predicate on X and p be a set over X. We define

$$MR\,p \;:=\; \{ x \in p \mid \forall y \in p.\, Ryx \to y = x \}$$

and call R **well-founded** if $MR\,p$ is inhabited whenever p is inhabited. Note that $\lambda xy.\bot$ is well-founded, and that every well-ordering is well-founded. We say that a binary predicate $\leq$ **extends** R if $x \leq y$ whenever Rxy.

Corollary 26 (Existence of Well-Ordered Extensions) Let X be an extensional type with a choice function. Then every well-founded predicate on X can be extended into a well-ordering of X.

Proof Let R be a well-founded predicate on X. From the given choice function we obtain a choice function γ such that $\gamma p \subseteq MR\,p$ for all sets p over X. This is possible since R is well-founded. By Theorem 25 we have a well-ordering $\leq$ of X that agrees with γ. Let Ryx. We show $y \leq x$ by contradiction. Suppose not $y \leq x$. By Linearity and excluded middle we have $x < y$. Since $\leq$ agrees with γ, we have $x \in \gamma(\neg S_x) \subseteq MR(\neg S_x)$. Since $y \in \neg S_x$ and Ryx, we have $y = x$. Contradiction. ∎

We return to the tower construction and show that Σ contains exactly the lower sets of the well-ordering $\leq_\eta$ of $\bigcup\Sigma$.

Fact 27 Let $p, q \in \Sigma$.

1. If $p \subset q$, then $\eta p \neq \eta q$.
2. If $\eta p = \eta q$, then $p = q$.

Proof Claim 2 follows with excluded middle and Linearity from Claim 1. To show Claim 1, let $p \subset q$. Then $\eta p \subseteq q$ by Fact 10. Suppose $\eta p = \eta q$. Then $\eta q = \emptyset$ and thus $\eta p = \emptyset$. Hence $p = \bigcup \Sigma$ by Fact 12. Contradiction since $p \subset q \in \Sigma$. ∎

Fact 28 $p \in \Sigma$ if and only if $p = \bigcup \Sigma$ or $p = \overline{x}$ for some $x \in \bigcup \Sigma$.

Proof The direction from right to left is obvious. For the other direction assume $p \in \Sigma$ and $p \neq \bigcup \Sigma$. By Fact 12 we have some $x \in \eta p \in \bigcup \Sigma$. By Fact 27 we have $p = \overline{x}$ if $\eta p = \eta \overline{x}$. Follows by Fact 20 and the uniqueness of η. ∎

Bourbaki [2] shows that Zermelo's theorem can be elegantly obtained from Zorn's lemma. A similar proof appears in Halmos [4]. Both proofs can be based equally well on Hausdorff's theorem.

7 General Towers

Our results for the specialized tower construction are not fully satisfactory. Intuition tells us that Σ should be well-ordered for every extension function, not just unique extension functions. However, in the proofs we have seen, uniqueness of the extension function is crucial. The search for a general proof led us to a generalized tower construction where the initial stage of a tower can be chosen freely (so far, the initial stage was always $\emptyset$). Now the lemmas and proofs can talk about all final segments of a tower, not just the full tower. This generality is needed for the inductive proofs to go through.

We make another abstraction step, which puts us in the setting of the Bourbaki-Witt fixed point theorem. Instead of taking sets as stages, we now consider towers whose stages are abstract points of some partially ordered type X. We assume an increasing function $f : X \to X$ to account for adjunction, and a subset-closed family S of sets over X having suprema to account for union. Given a starting point a, the tower for a is obtained by closing under f and suprema from S. We regain the concrete tower construction by choosing set inclusion as ordering and the family of all families of sets for S.

To understand the general tower construction, it is best to forget the specialized tower construction and start from the intuitions underlying the general construction. We drop all assumptions made so far except for excluded middle.

Assumption 29 Let X be a type and $\leq$ be a partial order on X. Moreover, let f be a function $X \to X$, S be a family over X, and $\sqcup$ be a function $\mathit{set}\, X \to X$ such that:

1. For all x: $x \le fx$ (f is **increasing**).
2. For all $p \subseteq q$: $p \in S$ if $q \in S$ (S is subset-closed).
3. For all $p \in S$: $\sqcup p \le x \leftrightarrow \forall z \in p.\, z \le x$ ($\sqcup$ yields suprema on S).

We call fx the **successor of x**, $\sqcup p$ the **join of p**, and the sets in S **admissible**.

We think of a tuple $(X, \le, \sqcup, S)$ satisfying the above assumptions as an S-*complete partial order* having suprema for all admissible sets.

Definition 30 We inductively define a binary predicate $\triangleright$ we call **reachability**:

$$\frac{}{x \triangleright x} \qquad \frac{x \triangleright y}{x \triangleright fy} \qquad \frac{p \in S \qquad p \text{ inhabited} \qquad \forall y \in p.\, x \triangleright y}{x \triangleright \sqcup p}$$

Informally, $x \triangleright y$ means that x can reach y by transfinite iteration of f. We define some helpful notations:

$$\begin{aligned} \Sigma_x &:= \{\, y \mid x \triangleright y \,\} && \textbf{tower for } x \\ x \bowtie y &:= x \triangleright y \lor y \triangleright x && x \textbf{ and } y \textbf{ connected} \\ x \triangleright p &:= p \in S \land p \text{ inhabited} \land \forall y \in p.\, x \triangleright y \end{aligned}$$

We will show that every tower is well-ordered by $\le$ and that the predicates $\le$ and $\triangleright$ agree on towers. With the notation $x \triangleright p$ we can write the join rule more compactly:

$$\frac{x \triangleright p}{x \triangleright \sqcup p}$$

Fact 31 If $x \in p \in S$, then $x \le \sqcup p$.

Fact 32 If $x \triangleright y$, then $x \le y$.

Proof By induction on $x \triangleright y$ exploiting the assumption that f is increasing. ∎

Lemma 33 (Strong Join Rule) The following rule is admissible for $\triangleright$.

$$\frac{p \in S \qquad p \text{ inhabited} \qquad \forall y \in p\ \exists z \in p.\, y \le z \land x \triangleright z}{x \triangleright \sqcup p}$$

Proof We have $\sqcup p = \sqcup(p \cap \Sigma_x)$ by the assumptions, Fact 31, and the subset-closedness of S. We also have $x \triangleright p \cap \Sigma_x$. Thus the conclusion follows with the join rule. ∎

Fact 34 (Successor) If $x \triangleright y$, then either $x = y$ or $fx \triangleright y$.

Proof By induction on $x \rhd y$. There is a case for every rule. The case for the first rule is trivial.

Successor. Let $x \rhd y$. We show $x = fy$ or $fx \rhd fy$. By the inductive hypotheses we have $x = y$ or $fx \rhd y$. The claim follows.

Join. Let $x \rhd p$. We show $x = \sqcup p$ or $fx \rhd \sqcup p$. Case analysis with excluded middle.

1. p unique. We have $\sqcup p = y$ for some $y \in p$ since p is inhabited. By the inductive hypothesis for $x \rhd y$ we have $x = y$ or $fx \rhd y$. The claim follows.
2. p not unique. We show $fx \rhd \sqcup p$ by Lemma 33. Let $y \in p$. We need some $z \in p$ such that $y \leq z$ and $x \rhd z$. By the inductive hypothesis for $x \rhd y$ we have two cases.
   a) $fx \rhd y$. The claim follows with $z := y$.
   b) $x = y$. We need some $z \in p$ such that $x \leq z$ and $fx \rhd z$. Since p is not unique, we have $z \in p$ different from x. By the inductive hypothesis for $x \rhd z$ we have $fx \rhd z$. The claim follows with $z := z$. ∎

Lemma 35 Let $x \bowtie \sqcup q$ for all $x \in p$. Then either $\sqcup p \leq \sqcup q$ or $\sqcup q \rhd \sqcup p$.

Proof Case analysis using excluded middle.

1. $\sqcup q \rhd x$ for some $x \in p$. We show $\sqcup q \rhd \sqcup p$ with Lemma 33. Let $y \in p$. We need some $z \in p$ such that $y \leq z$ and $\sqcup q \rhd z$. By assumption we have $y \bowtie \sqcup q$. If $y \rhd \sqcup q$, the claim follows with $z := x$ since $y \leq x$ by Fact 32. If $\sqcup q \rhd y$, the claim follows with $z := y$.
2. $\sqcup q \rhd x$ for no $x \in p$. We show $\sqcup p \leq \sqcup q$. Let $y \in p$. We show $x \leq \sqcup q$. By assumption we have $x \bowtie \sqcup q$. If $x \rhd \sqcup q$, the claim follows by Fact 32. If $\sqcup q \rhd x$, we have a contradiction. ∎

Lemma 36 Let $a \rhd x$ and $a \rhd y$. Then $x \bowtie y$.

Proof By nested induction on $a \rhd x$ and $a \rhd y$. The cases where $x = a$ or $y = a$ are trivial. The cases were $x = fx'$ or $y = fy'$ are straightforward with Fact 34. Now only the case where x and y are both joins remains.

We have $a \rhd \sqcup p$, $a \rhd \sqcup q$, $a \rhd p$, and $a \rhd q$. The inductive hypotheses give us $x \bowtie \sqcup q$ for all $x \in p$ and $x \bowtie \sqcup p$ for all $x \in q$. By Lemma 35 we have $\sqcup p \leq \sqcup q$ or $\sqcup q \rhd \sqcup p$ and also $\sqcup q \leq \sqcup p$ or $\sqcup p \rhd \sqcup q$. Thus $\sqcup p \bowtie \sqcup q$ by antisymmetry of $\leq$. ∎

Theorem 37 (Coincidence and Linearity) Let $x, y \in \Sigma_a$. Then:

1. $x \leq y$ iff $x \rhd y$.
2. $x \leq y$ or $y \leq x$.

Proof Follows with Facts 32 and Lemma 36. ∎

Lemma 38 Let $a \triangleright b = fb$. Then $x \triangleright b$ whenever $a \triangleright x$.

Proof We show $x \triangleright b$ induction on $a \triangleright x$. For the first rule the claim is trivial.

Successor. Let $a \triangleright x$. We show $fx \triangleright b$. We have $x \triangleright b$ by the inductive hypothesis. The claim follows with Fact 34.

Join. Let $a \triangleright p$. We show $\sqcup p \triangleright b$. By Theorem 37 it suffices to show $\sqcup p \leq b$. Let $x \in p$. We show $x \leq b$. Follows by the inductive hypothesis. ∎

Theorem 39 (Fixed Point)

1. $fx = x$ and $x \in \Sigma_a$ iff x is the greatest element of Σ_a.
2. If $\Sigma_a \in S$, then $\sqcup\Sigma_a$ is the greatest element of Σ_a.

Proof Claim (1) follows with Lemma 38 and Fact 32. For Claim (2) let $\Sigma_a \in S$. Then $a \triangleright \Sigma_a$. Thus $a \triangleright \sqcup\Sigma_a$ and $\sqcup\Sigma_a \in \Sigma_a$. The claim follows with Fact 31. ∎

Fact 40 (Successor)
Let $x, y \in \Sigma_a$ and $x \leq y \leq fx$. Then either $x = y$ or $y = fx$.

Proof By Coincidence we have $x \triangleright y$. By Fact 34 we have $x = y$ or $fx \triangleright y$. If $x = y$, we are done. Let $fx \triangleright y$. By Coincidence we have $fx \leq y$. Thus $y = fx$ by the assumption and antisymmetry of $\leq$. ∎

Fact 41 (Join) Let $x \in \Sigma_a$, $p \subseteq \Sigma_a$, $p \in S$, and $x < \sqcup p$. Then there exists $y \in p$ such that $x < y$.

Proof By contradiction. Suppose $x \not< y$ for all $y \in p$. By Linearity we have $y \leq x$ for all $y \in p$. Thus $\sqcup p \leq x$ and therefore $x < x$. Contradiction. ∎

8 Well-Ordering of General Towers

We already know that every tower Σ_a is linearly ordered by $\leq$ (Theorem 37). We now show that every tower is well-ordered by $\leq$. To do so, we establish an induction principle for towers, from which we obtain the existence of least elements. We use an inductive definition to establish the induction principle.

Definition 42 For every a in X we inductively define a set I_a:

$$\frac{x \in \Sigma_a \qquad \forall y \in \Sigma_a.\, y < x \to y \in I_a}{x \in I_a}$$

Lemma 43 (Induction) $\Sigma_a \subseteq I_a$ and $I_a \subseteq \Sigma_a$.

Proof $I_a \subseteq \Sigma_a$ is obvious from the definition of I_a. For the other direction, let $x \in \Sigma_a$. We show $x \in I_a$ by induction on $a \rhd x$. We have three cases. If $x = a$, the claim is obvious.

Sucessor. Let $a \rhd x$. We show $fx \in I_a$. Let $y \in \Sigma_a$ such that $y < fx$. We show $y \in I_a$. We have $x \in I_a$ by the inductive hypothesis. By Linearity and excluded middle we have three cases. If $x < y$, we have a contradiction by Fact 40. If $x = y$, the claim follows from $x \in I_a$. If $y < x$, the claim follows by inversion of $x \in I_a$.

Join. Let $a \rhd p$. We show $\sqcup p \in I_a$. Let $y \in \Sigma_a$ such that $y < \sqcup p$. We show $y \in I_a$. Since $y < \sqcup p$, we have $z \in p$ such that $y < z$ by Fact 41. By the inductive hypothesis we have $z \in I_a$. The claim follows by inversion. ∎

Theorem 44 (Well-Ordering) Σ_a is well-ordered by $\leq$.

Proof Let $x \in p \subseteq \Sigma_a$. Then $x \in I_a$ by Lemma 43. We show by induction on $x \in I_a$ that $L(\leq)p$ is inhabited. If $x \in L(\leq)p$, we are done. Otherwise, we have $y \in p$ such that $y < x$ by Linearity (Theorem 37). The claim follows by the inductive hypothesis. ∎

We can now prove a generalized version of the Bourbaki-Witt theorem.

Theorem 45 (Bourbaki-Witt, Generalized) Let X be a type, $\leq$ be a partial order of X, and $\sqcup$ be a function $\mathit{set}\, X \to X$ such that $\sqcup p$ is the supremum of p whenever p is well-ordered by $\leq$. Let f be an increasing function $X \to X$ and a be an element of X. Then f has a fixed point above a.

Proof Let S be the family of all sets over X that are well-ordered by $\leq$. Then all assumptions made so far are satisfied. By Theorem 44 we know that Σ_a is well-ordered by $\leq$. Thus $\Sigma_a \in S$. The claim follows with Theorem 39. ∎

The general tower construction can be instantiated so that it yields the special tower construction considered in the first part of the paper. Based on this instantiation, the theorems of Hausdorff and Zermelo can be shown as before. For the generalized version of Hausdorff's theorem (extension of a given chain), the fact that the general construction yields a tower for every starting point provides for a simpler proof.

9 Final Remarks

We have studied the tower construction and some of its applications in classical type theory (i.e., excluded middle and impredicative universe of propositions). The tower construction is a transfinite construction from set theory used in the proofs of the theorems of Zermelo, Hausdorff, and Bourbaki and Witt. The general form of the tower construction in set theory was identified by Bourbaki [2] in 1949.

Translating the tower construction and the mentioned results from set theory to classical type theory is not difficult. There is no need for an axiomatized type of sets. The sets used in the set-theoretic presentation can be expressed as types and as predicates (both forms are needed). The tower construction can be naturally expressed with an inductive definition.

We have studied a specialized and a general form of the tower construction, both expressed with an inductive definition. The specialized form enjoys straightforward proofs and suffices for the proofs of the theorems of Zermelo and Hausdorff. The general form applies to a partially ordered type and is needed for the proof of the Bourbaki-Witt fixed point theorem. Our proofs of the properties of the tower construction are different from the proofs in the set-theoretic literature in that they make substantial use of induction. The proofs in the set-theoretic literature have less structure and use proof by contradiction in place of induction (with the exception of Felscher [3]).

There are two prominent unbounded towers in axiomatic set theory: The class of ordinals and the cumulative hierarchy. The cumulative hierarchy is usually obtained with transfinite recursion on ordinals. If we consider an axiomatized type of sets in type theory, the general tower construction of this paper yields a direct construction of the cumulative hierarchy (i.e., a construction not using ordinals). Such a direct construction of the cumulative hierarchy will be the subject of a forthcoming paper.[1]

We have obtained Hausdorff's and Zermelo's theorem under the assumption that the underlying base type is extensional. This assumption was made for simplicity and can be dropped if one works with extensional choice functions and equivalence of sets.

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[1] Our initial motivation for studying the general tower construction was the direct construction of the cumulative hierarchy in axiomatic set theory. Only after finishing the proofs for the general tower construction in type theory, we discovered Bourbaki's marvelous presentation [2] of the tower construction in set theory.

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