Analytic Tableaux for Higher-Order Logic with Choice

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Introduction: (Some) Higher-Order Provers

Automated	Interactive
Isabelle-HOL	Isabelle-HOL
TPS	HOL family:
	HOL4
	HOL-Light
	ProofPower
Simple Type Theory	Even More Simple Types
+ Logical Constants	+ Logical Constants
	+ Extensionality
	+ Primitive Equality
	+ Choice
	+ Infinity

Andrews, Paulson, Nipkow, Gordon, . . .

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If p is nonempty, then εp is in p.

$$\forall px.px \longrightarrow p(\varepsilon p)$$

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Similar tableau rule: Split into p empty or $p(\varepsilon p)$.

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Mints: Because εp is a subterm. Can we further restrict?

Outline

- Higher-Order Logic with Choice (ε)
- Higher-Order Tableau
- The Choice Rule
- Restrictions on Instantiations
- Examples

Higher Order Logic

Simple Types σ, τ:

o (propositions), ι (individuals), $\sigma \rightarrow \tau$ (functions)

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Variables x of each type.

Terms s, t :

$$x \mid c \mid (\lambda x.s) \mid (st)$$

Logical constants $c \ldots$ (next slide)

$[s]$ - normal form of s

Logical Constants

- $\bot : o$
- $\neg : o \longrightarrow o$
- $\vee : o \longrightarrow o \longrightarrow o$
- $=_\sigma : \sigma \longrightarrow \sigma \longrightarrow o$
- $\forall_\sigma : (\sigma \longrightarrow o) \longrightarrow o$

Logical Constants

- $\bot : o$
- $\neg : o \to o$
- $\vee : o \to o \to o$
- $=_\sigma : \sigma \to \sigma \to o$
- $\forall_\sigma : (\sigma \to o) \to o$
- $\varepsilon_\sigma : (\sigma \to o) \to \sigma$ - choice function on σ
  Notation: Write $\varepsilon x.s$ for $\varepsilon(\lambda x.s)$

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- $* : \iota$ - a default element of the nonempty type ι

Example 2

Example 2: Fix $p : \iota \to o$. Can we have $\forall_o q.\varepsilon p \neq \varepsilon x.q$?

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Tableau procedure will decide whether or not this is satisfiable.

Higher Order Tableau

Brown, Smolka [LMCS 2010]
(Complete for Henkin models without Choice)

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A few unsurprising rules...

$$\mathcal{T}_{\neg}\ \frac{s, \neg s}{\bot} \qquad \mathcal{T}_{\neq}\ \frac{s \neq_{\iota} s}{\bot} \qquad \mathcal{T}_{\neg\neg}\ \frac{\neg\neg s}{s} \qquad \mathcal{T}_{\vee}\ \frac{s \vee t}{s \mid t}$$

$$\mathcal{T}_{\neg\vee}\ \frac{\neg(s \vee t)}{\neg s, \neg t}$$

Higher Order Tableau

Brown, Smolka [LMCS 2010]
(Complete for Henkin models without Choice)
Mating and decomposition...

$$\mathcal{T}_{\text{MAT}} \quad \frac{\delta s\,,\ \neg\delta t}{s \neq t} \qquad\qquad \mathcal{T}_{\text{DEC}} \quad \frac{\delta s \neq_{\iota} \delta t}{s \neq t}$$

δ either a variable or an ε (also, for arity > 1)

Higher Order Tableau

Brown, Smolka [LMCS 2010]
(Complete for Henkin models without Choice)
...and more rules...extensionality, equality

$$\mathcal{T}_{\mathrm{CON}} \quad \frac{s =_\iota t \,,\, u \neq_\iota v}{s \neq u \,,\, t \neq u \mid s \neq v \,,\, t \neq v} \qquad \mathcal{T}_{\mathrm{BE}} \quad \frac{s \neq_o t}{s \,,\, \neg t \mid \neg s \,,\, t}$$

$$\mathcal{T}_{\mathrm{BQ}} \quad \frac{s =_o t}{s \,,\, t \mid \neg s \,,\, \neg t}$$

$$\mathcal{T}_{\mathrm{FE}} \quad \frac{s \neq_{\sigma\tau} t}{\neg[\forall x.sx = tx]} \quad x \notin \mathcal{V}s \cup \mathcal{V}t$$

$$\mathcal{T}_{\mathrm{FQ}} \quad \frac{s =_{\sigma\tau} t}{[\forall x.sx = tx]} \quad x \notin \mathcal{V}s \cup \mathcal{V}t$$

The Choice Rule

$$\mathcal{T}_\varepsilon \quad \frac{}{[\forall x.\neg(sx)] \;\mid\; [s(\varepsilon s)]} \; \varepsilon s \text{ accessible}, x \notin \mathcal{V}s$$

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o: εs or $\neg\varepsilon s$ on the branch

$\sigma_1 \to \cdots \to \sigma_n \to o$: $(\varepsilon s)u_1 \cdots u_n$ or its negation on the branch

Restrictions on Instantiations

$$\mathcal{T}_{\forall} \quad \frac{\forall_\sigma s}{[st]} \quad t \in \mathcal{U}_\sigma$$

Restrict instantiations in the $\mathcal{T}_{\forall}$ rule based to $\mathcal{U}_\sigma$ on the type:

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Finitely many!

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  Finitely many!
- $\mathcal{U}_o$ Only $\bot$ and $\neg\bot$ (false and true)
  Finitely many!
- $\mathcal{U}_{\sigma\to\tau}$ Only normal terms using variables free on the branch
  Infinitely many, of course.

If only quantifiers at o and ι, the procedure sometimes terminates.

Example 1

$$px$$
$$\neg p(\varepsilon p)$$

Example 1

$$p x$$
$$\neg p(\varepsilon p)$$

Example 1

$$px$$
$$\neg p(\varepsilon p)$$
$$x \neq_\iota \varepsilon p$$

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Example 1

εp accessible - Choice Rule Activated

$$px$$
$$\neg p(\varepsilon p)$$
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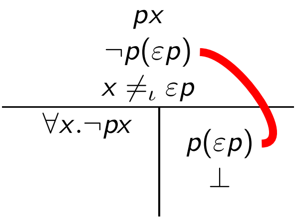
Example 1

$$px$$
$$\neg p(\varepsilon p)$$
$$x \neq_\iota \varepsilon p$$

$\forall x.\neg px$	$p(\varepsilon p)$

Example 1


px
$\neg p(\varepsilon p)$
$x \neq_\iota \varepsilon p$
$\forall x.\neg px$
$p(\varepsilon p)$
$\bot$

Example 1

x accessible - Legal Instantiation for $\forall$

$$px$$
$$\neg p(\varepsilon p)$$
$$x \neq_\iota \varepsilon p$$

$\forall x.\neg px$	$p(\varepsilon p)$
	$\bot$

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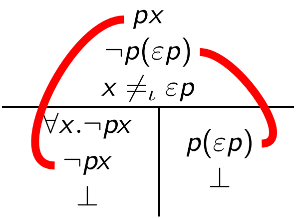

px
$\neg p(\varepsilon p)$
$x \neq_\iota \varepsilon p$
$\forall x. \neg px$
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$\bot$
$p(\varepsilon p)$
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Example 2

$$\forall_o q.\varepsilon p \neq \varepsilon x.q$$

Example 2

Instantiate with $\bot$ and $\neg\bot$

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Example 2

Decompose

$$\forall_o q. \varepsilon p \neq \varepsilon x.q$$
$$\varepsilon p \neq \varepsilon x.\bot$$
$$\varepsilon p \neq \varepsilon x.\neg\bot$$

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Decompose

$$\forall_o q.\varepsilon p \neq \varepsilon x.q$$
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px $\neg\bot$		$\neg px$ $\bot$
py $\neg\neg\bot$ $\bot$	$\neg py$	

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<table>
<tr><td colspan="2">px
$\neg\bot$</td><td rowspan="2">$\neg px$
$\bot$</td></tr>
<tr><td>py
$\neg\neg\bot$
$\bot$</td><td>$\neg py$
$x \neq y$</td></tr>
</table>

Example 2

Accessible Choice Terms: εp, $\varepsilon x.\bot$, $\varepsilon x.\neg\bot$

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Branch 1:

$$px$$
$$\neg\bot$$

- Branch 1.1: py, $\neg\neg\bot$, $\bot$
- Branch 1.2: $\neg py$, $x \neq y$

Branch 2:

$$\neg px$$
$$\bot$$

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$$x \neq y$$

$$\forall x.\neg px$$
$$\vdots$$

$$p(\varepsilon p)$$
$$\varepsilon p \neq y$$

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Example 2

Evident (Hintikka) Set: Satisfiable

$$\forall_o q.\varepsilon p \neq \varepsilon x.q$$
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Branch 1.2.2:

$$p(\varepsilon p)$$
$$\varepsilon p \neq y$$

Branch 1.2.2.1:

$$\forall x.\neg\bot$$

Branch 1.2.2.2:

$$\bot$$

Branch 2:

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$$\bot$$

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  (compatible sets of discriminating terms - those used in disequations) - Brown, Smolka [LMCS 2010]

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- and interpret ι by discriminants (compatible sets of discriminating terms - those used in disequations) - Brown, Smolka [LMCS 2010]
- and interpret Choice using a definition similar to Mints [JSL 1999]

Completeness (2): Possible Values

- Given: Branch A that is not refutable.
- Extend to E satisfying certain properties. ($A \subseteq E$)
- Let X be the free variables in E.
- Define $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda_\sigma^X \times \mathcal{D}_\sigma$ by induction on types.

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- Interpret variables x such that $x \triangleright \mathcal{I}x$.
- Interpret logical constants c appropriately and ensure $c \triangleright \mathcal{I}c$.
- Result will be a Henkin model of E.

Completeness (3): Forall

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda_\sigma^X \times \mathcal{D}_\sigma$
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Completeness (3): Forall

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda_\sigma^X \times \mathcal{D}_\sigma$
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- Interpretation of $\forall_\sigma$ is clear:

$$(\mathcal{I}\forall_\sigma)(f) = \begin{cases} 1 & \text{if } fa = 1 \text{ for all } a \\ 0 & \text{otherwise} \end{cases}$$

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- Easy to check $\forall_\sigma \triangleright \mathcal{I}\forall_\sigma$.

Completeness (4): Choice

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda^X_\sigma \times \mathcal{D}_\sigma$
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Completeness (4): Choice

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda_\sigma^X \times \mathcal{D}_\sigma$
- $\mathcal{D}_o = \{0, 1\}$ (false and true)
- To interpret ε_σ, let $f \in \mathcal{D}_{\sigma \to o}$.

Completeness (4): Choice

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda_\sigma^X \times \mathcal{D}_\sigma$
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- To interpret ε_σ, let $f \in \mathcal{D}_{\sigma \to o}$.
- First attempt:

$$(\mathcal{I}\varepsilon_\sigma)(f) = \begin{cases} \text{some } b & \text{such that } fb = 1 \text{ if such a } b \text{ exists.} \\ \text{some } a & \text{otherwise.} \end{cases}$$

Completeness (4): Choice

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda^X_\sigma \times \mathcal{D}_\sigma$
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- To interpret ε_σ, let $f \in \mathcal{D}_{\sigma\to o}$.
- First attempt:

$$(\mathcal{I}\varepsilon_\sigma)(f) = \begin{cases} \text{some } b & \text{such that } fb = 1 \text{ if such a } b \text{ exists.} \\ \text{some } a & \text{otherwise.} \end{cases}$$

- Problem: Cannot ensure $\varepsilon_\sigma \triangleright \mathcal{I}\varepsilon_\sigma$

Completeness (4): Choice

- $\mathcal{D}_\sigma$ and $\triangleright_\sigma \subseteq \Lambda^X_\sigma \times \mathcal{D}_\sigma$
- $\mathcal{D}_o = \{0, 1\}$ (false and true)
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- $\mathcal{I}\varepsilon_\sigma$ is a choice function, and
- $\varepsilon_\sigma \triangleright \mathcal{I}\varepsilon_\sigma$.

Conclusion

- Directed, Cut-Free, Ground Tableau System for HOL (Brown, Smolka [LMCS 2010])
- Extended to include Choice (Backes, Brown [2010])
- Restricted Instantiations (Backes, Brown [2010])
- Similar techniques work for if-then-else and description (Backes [2010])
- Implementation: Satallax (Tableau + MiniSAT to search, competing in CASC)
- Can be used to determine both Unsatisfiability *and* (sometimes) Satisfiability.