Weakly Monotonic Propagators

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Reality check

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Beautiful model:

Propagators are monotonic!

closure operators, unique fixpoints, predictable behavior,
more constraints means less search

[Saraswat et al, Benhamou et al, Van Hentenryck et al, Apt, ...]

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Propagators are not monotonic!

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Propagators are monotonic! **Propagators are not monotonic!**

- Realistic formal model
  - establish properties of propagators
  - minimally restrictive (maximally realistic)
- Consequences
  - what does propagation compute?
  - does recomputation work?

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A Model

Propagation

Variables $x, y, z \in X$

Values V

Domains $d(x) \subseteq V$

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- **monotonicity is crucial**

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Non-monotonic propagators

- Some constraints are NP-hard to propagate
  - Hamiltonian circuit, knapsack, scheduling
- Approximative propagation algorithms are often non-monotonic, e.g.
  - consider some instead of all subsets
  - consider only small domains
  - randomize
  - start graph algorithm at one instead of all nodes

Example: Hamiltonian circuit

`circuit(`x_1`,...,`x_n`)`

graph with edges `i→`x_i

has single cycle

covering all nodes

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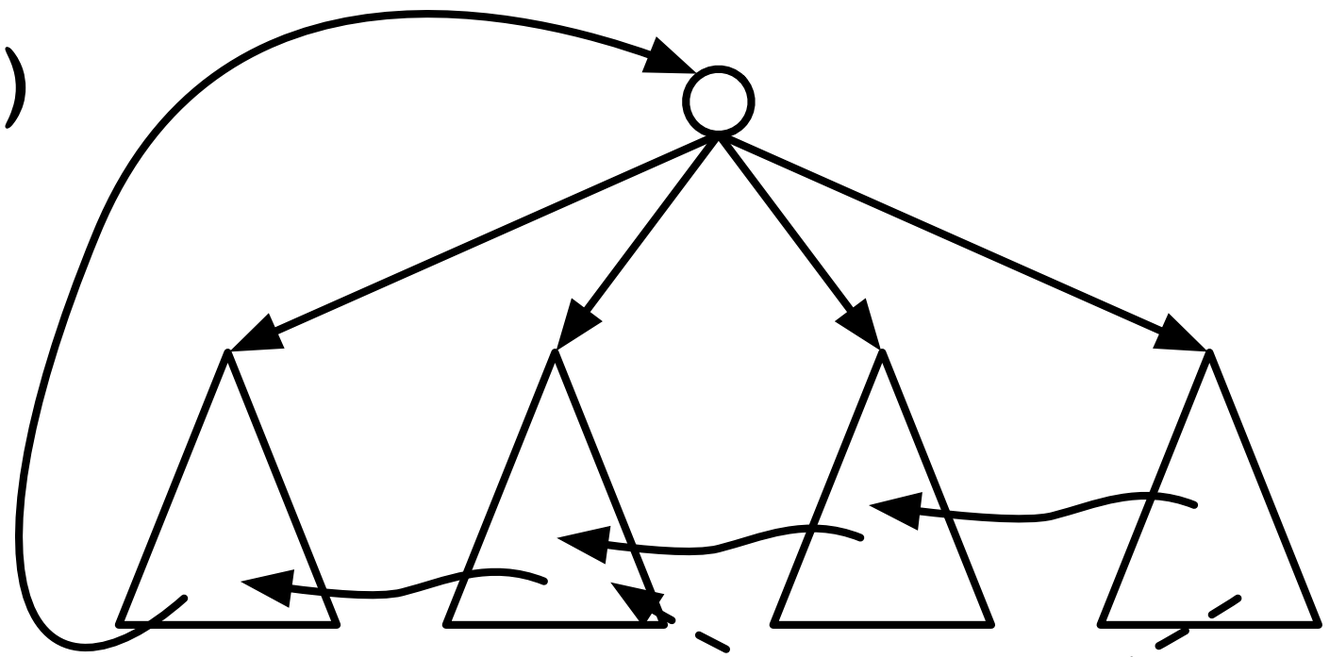

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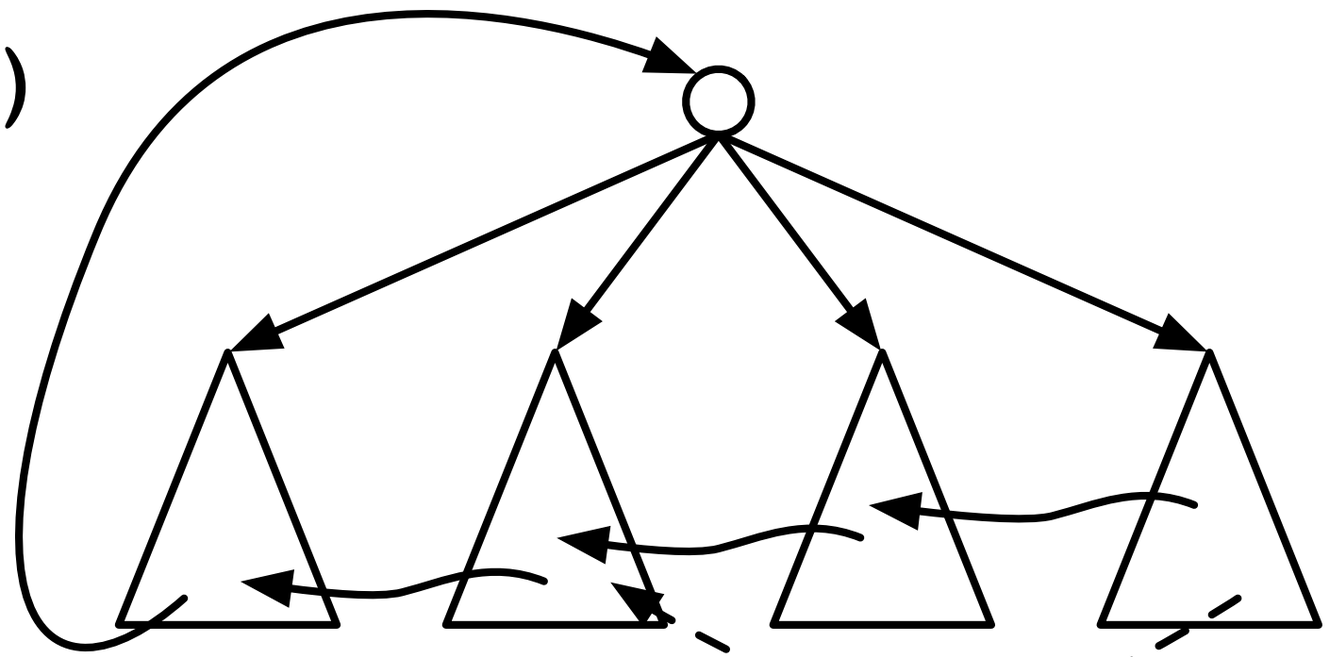

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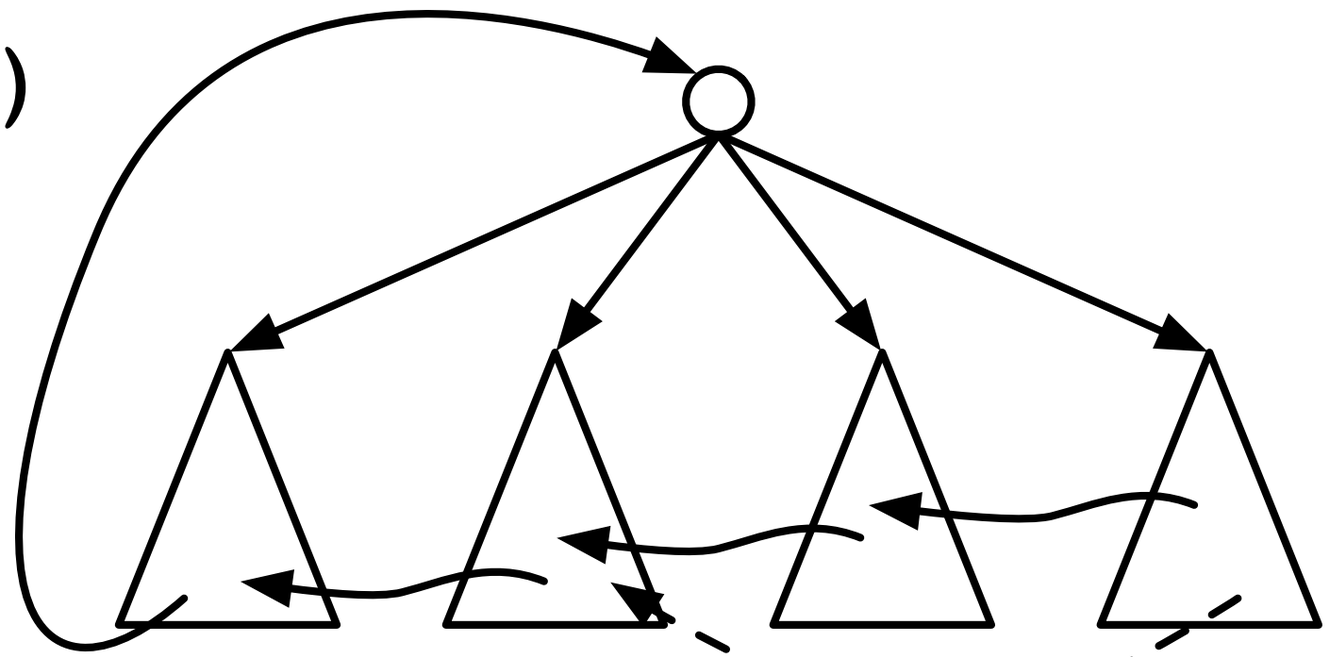

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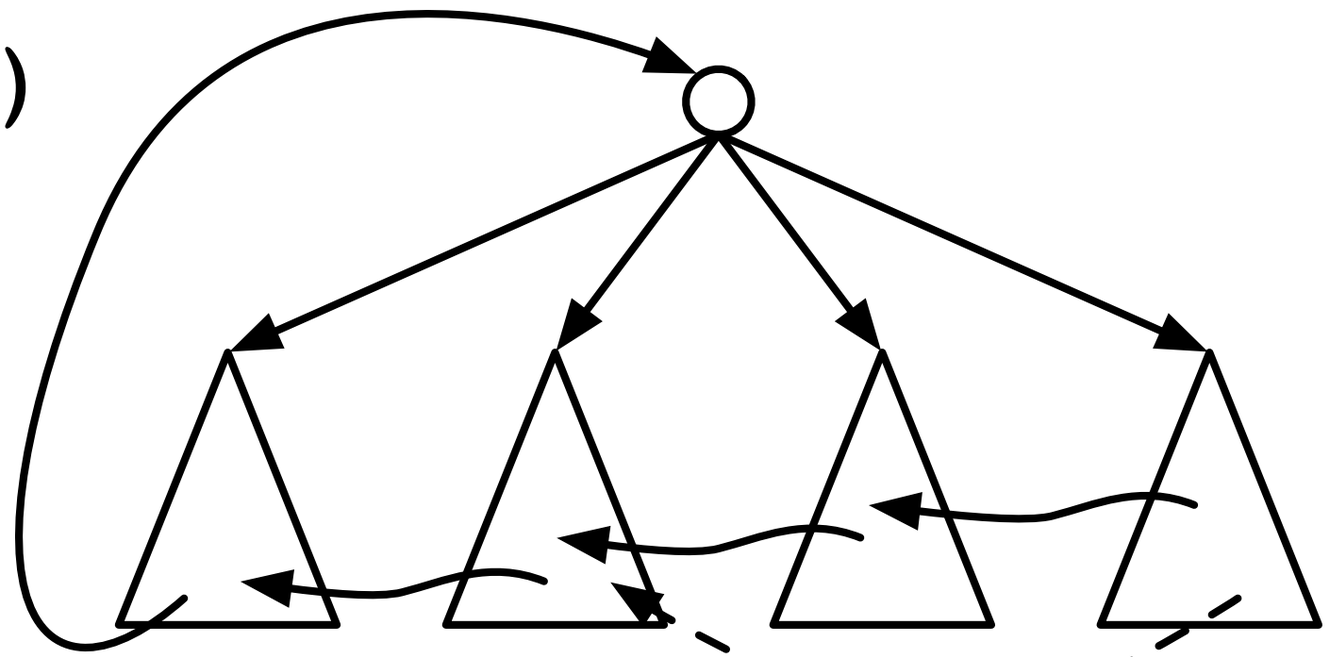

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- DFS start node arbitrary, thus **non-monotonic**

Example: Hamiltonian circuit

circu

graph v

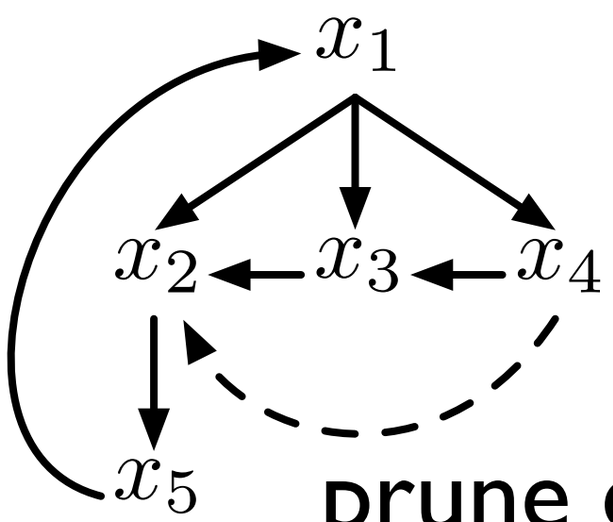
has sing

coverin

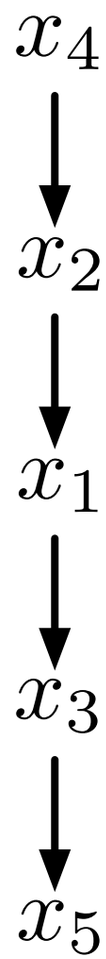
- chec
- prun
- fix e
- DFS

$$x_1 \in \{2,3,4\} \quad x_2 \in \{1,5\} \quad x_3 \in \{1,2,5\}$$

$$x_4 \in \{2,3\} \quad x_5 \in \{1,2\}$$



prune edge



no pruning

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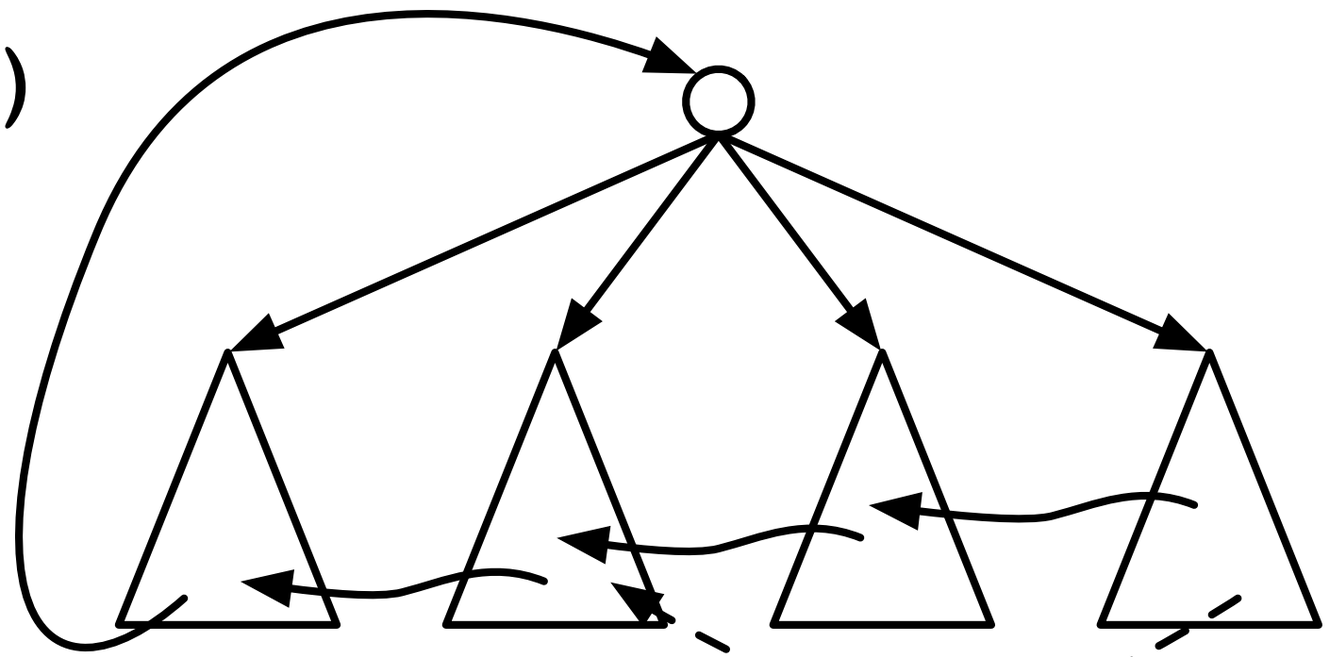

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A Realistic Model

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Assignments $a \in \mathrm{Asn} = X \to V$ $a(x) \in V$

Constraints $c \subseteq \mathrm{Asn}$

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$$
\begin{array}{rcll}
X & = & \{x, y, z\} & \\
V & = & \{1, 2, 3, 4\} & \\
d & = & \{x \mapsto V, y \mapsto V, z \mapsto V\} & \\
c_1 & = & \{a \in \mathrm{Asn} \mid a(x) \neq a(y) \wedge & \mathrm{distinct}(x, y, z) \\
 & & \quad a(x) \neq a(z) \wedge & \\
 & & \quad a(y) \neq a(z)\} & \\
c_2 & = & \{a \in \mathrm{Asn} \mid a(x) + a(y) = a(z)\} & x + y = z
\end{array}
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- **must prune *consistently*:**

$$\forall a \in d : \;\; a \notin p(d) \Rightarrow a \notin c_p$$

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- this is **weak** monotonicity

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Propagators $\{a\} \subseteq d \Rightarrow p(\{a\}) \subseteq p(d)$ (weakly m.)

$p(d)(x) \subseteq d(x)$ (contracting) ✔

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closure operators

Knaster-Tarski ⇒ weakest mutual fixpoint exists

$p(d) \subseteq dom(c_p \cap d)$ (complete)

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Knaster-Tarski **does not apply**

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- "weaker than GAC but not weaker than FC"
- **solution space is invariant**

Fixpoint Issues

- All depends on propagator order:
  - individual fixpoints (may not even be comparable!)
  - the shape of the search tree
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- Adding constraints can yield bigger search trees!
- **C'est la vie**

  (also happens with randomized heuristics, restarts, or parallel search)

Minimal model

$$p(d)(x) \subseteq d(x)$$ (contracting)

$$\{a\} \subseteq d \Rightarrow p(\{a\}) \subseteq p(d)$$ (weakly monotonic)

Any contracting function can be made weakly monotonic by composition with a **checker!**

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Model is minimally restrictive

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Randomized Propagation

- W
- Pr
- Ite
  fix
- Bu
- **W**

Circuit: random vs. fixed heuristic

runtime in % (smaller=better)

Knights 18	**17.0%**
Knights 20	98.1%
Knights 22	**1.1%**
Knights 24	**7.2%**
TSP br17	102.3%
TSP ftv33	97.1%

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Non-monotonic search

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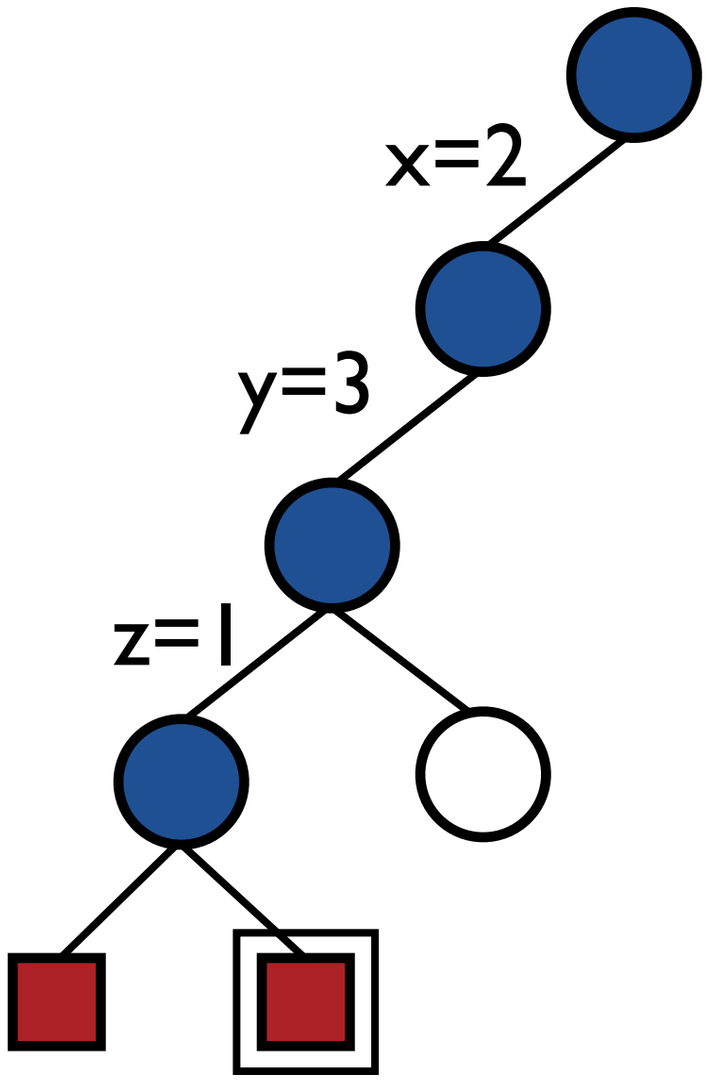

- record undo information on a *trail*
- on backtrack, restore ancestor state using undos
- for every node in the tree, compute exactly one fixpoint
- **supports non-monotonic propagation without change**

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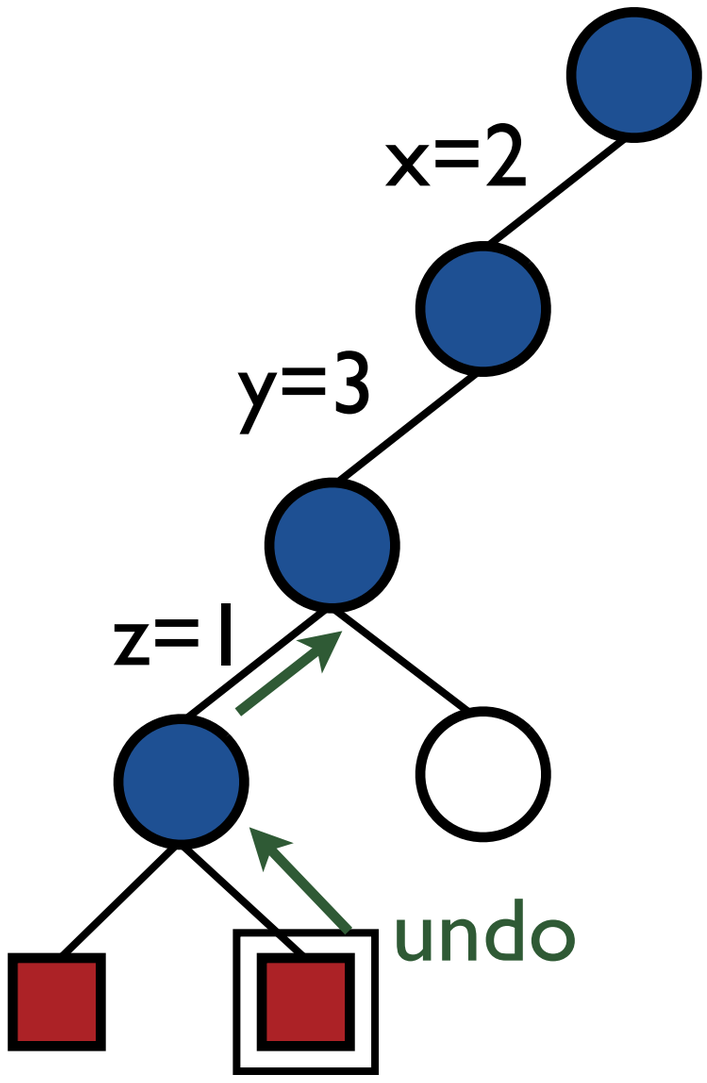

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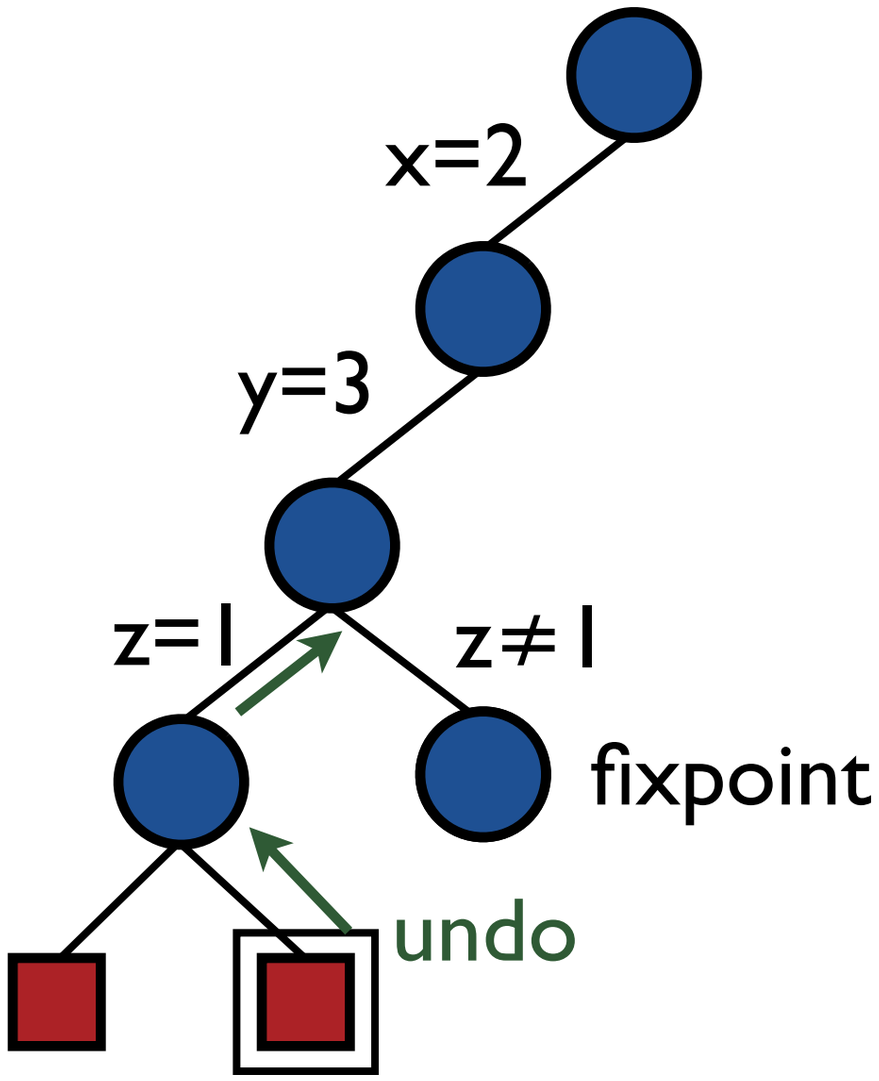

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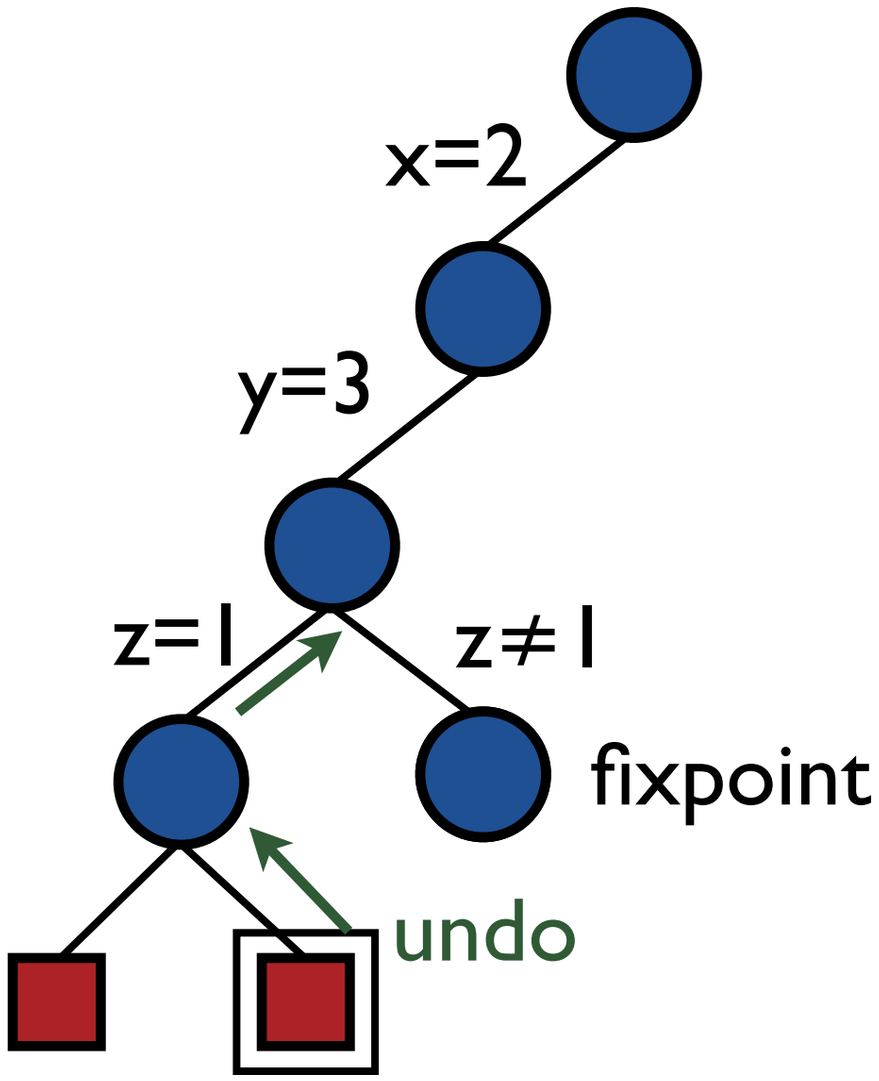

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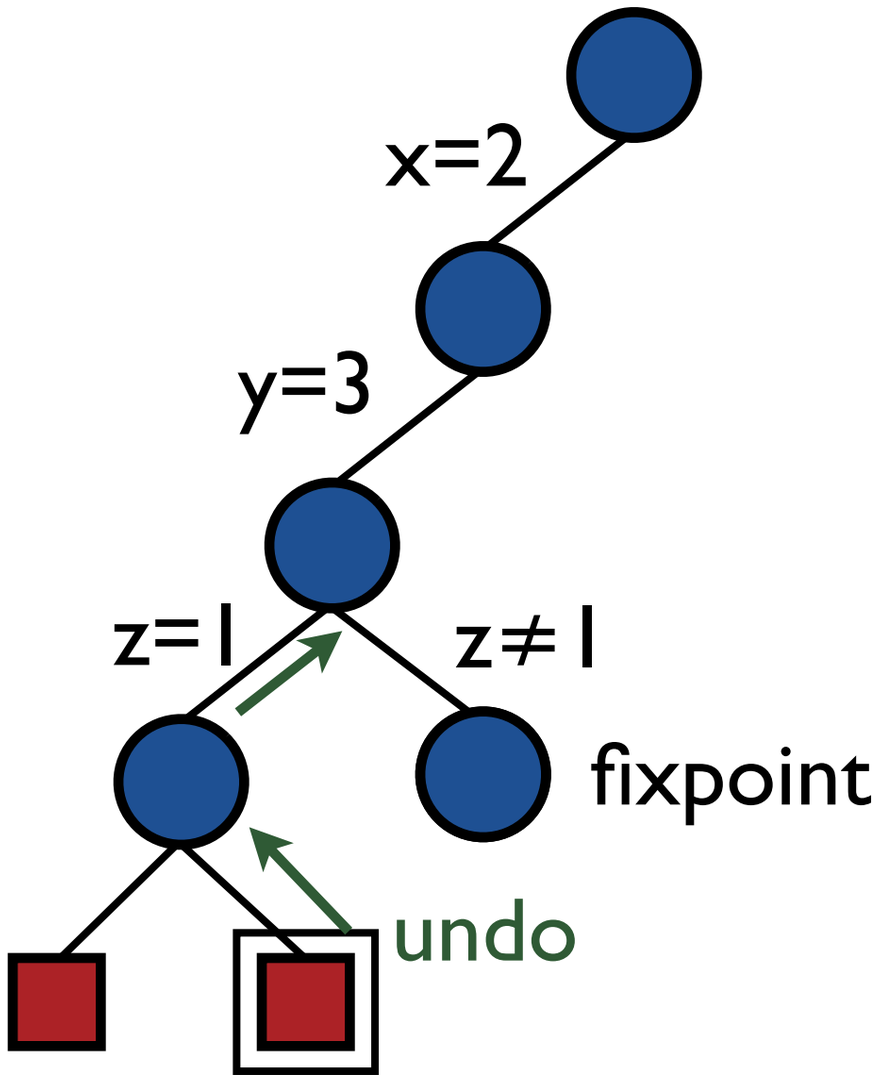

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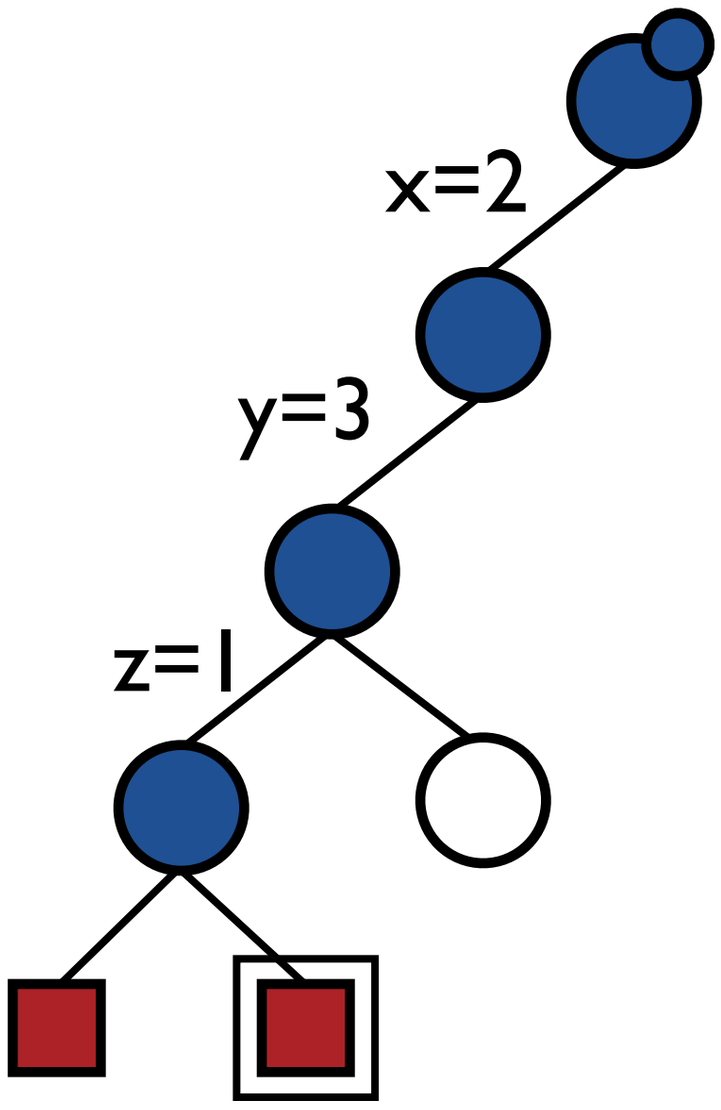

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- **does not work with non-monotonic propagators!**

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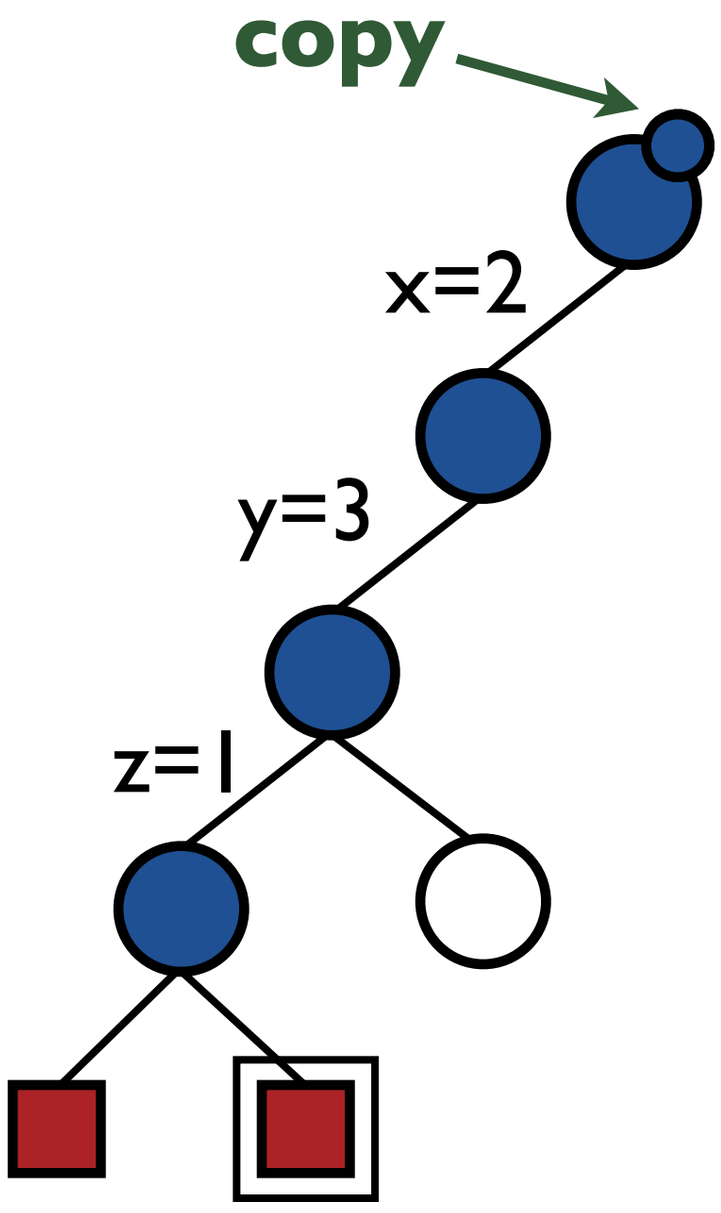

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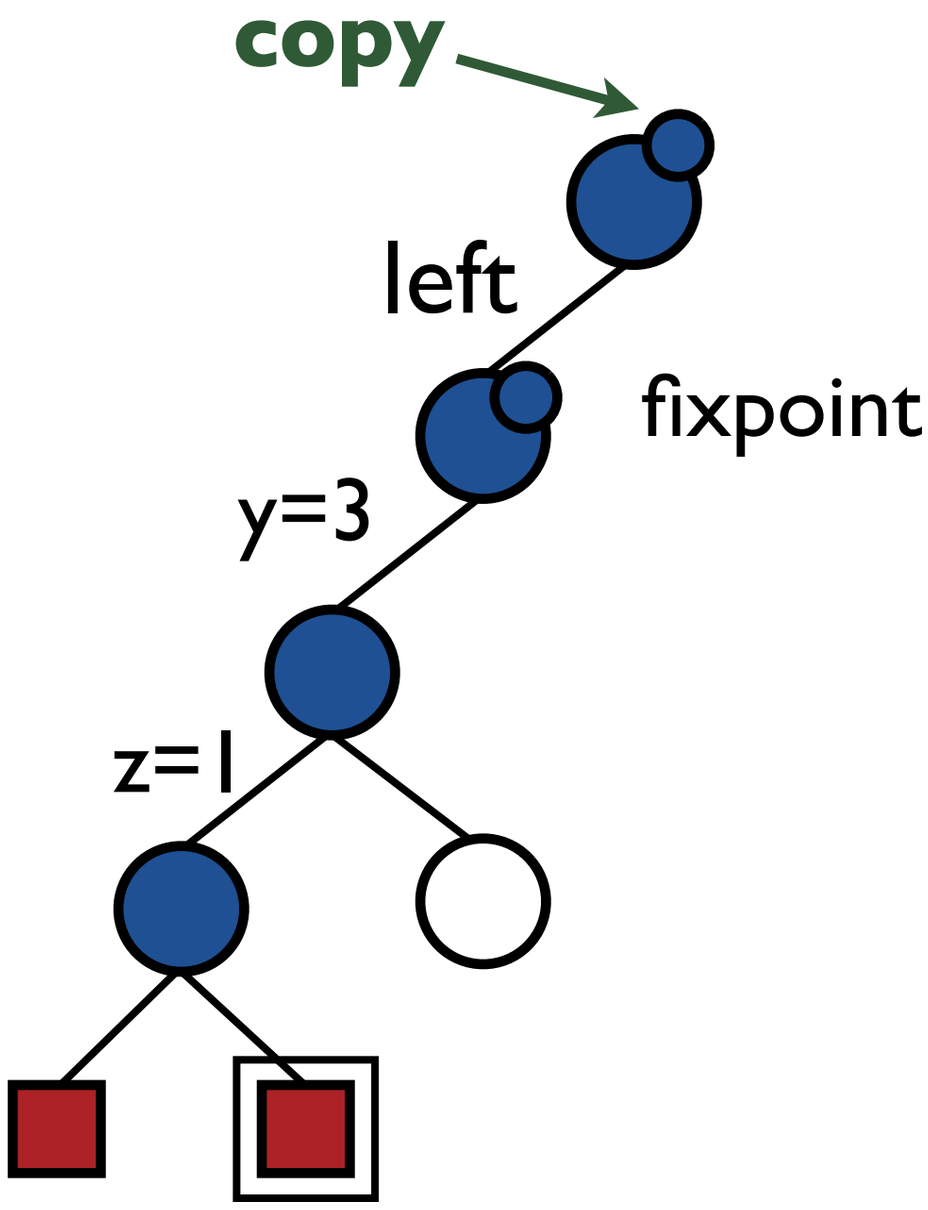

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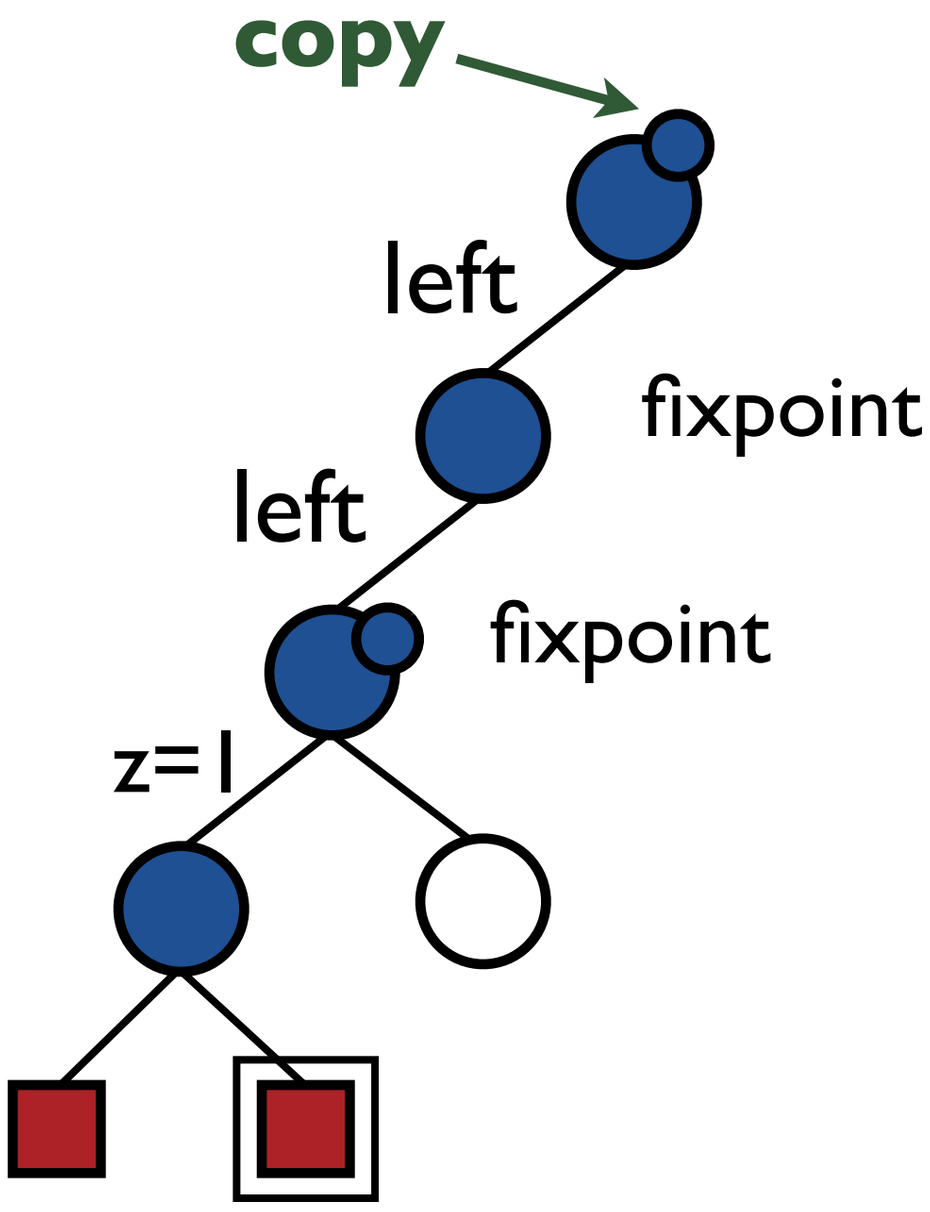

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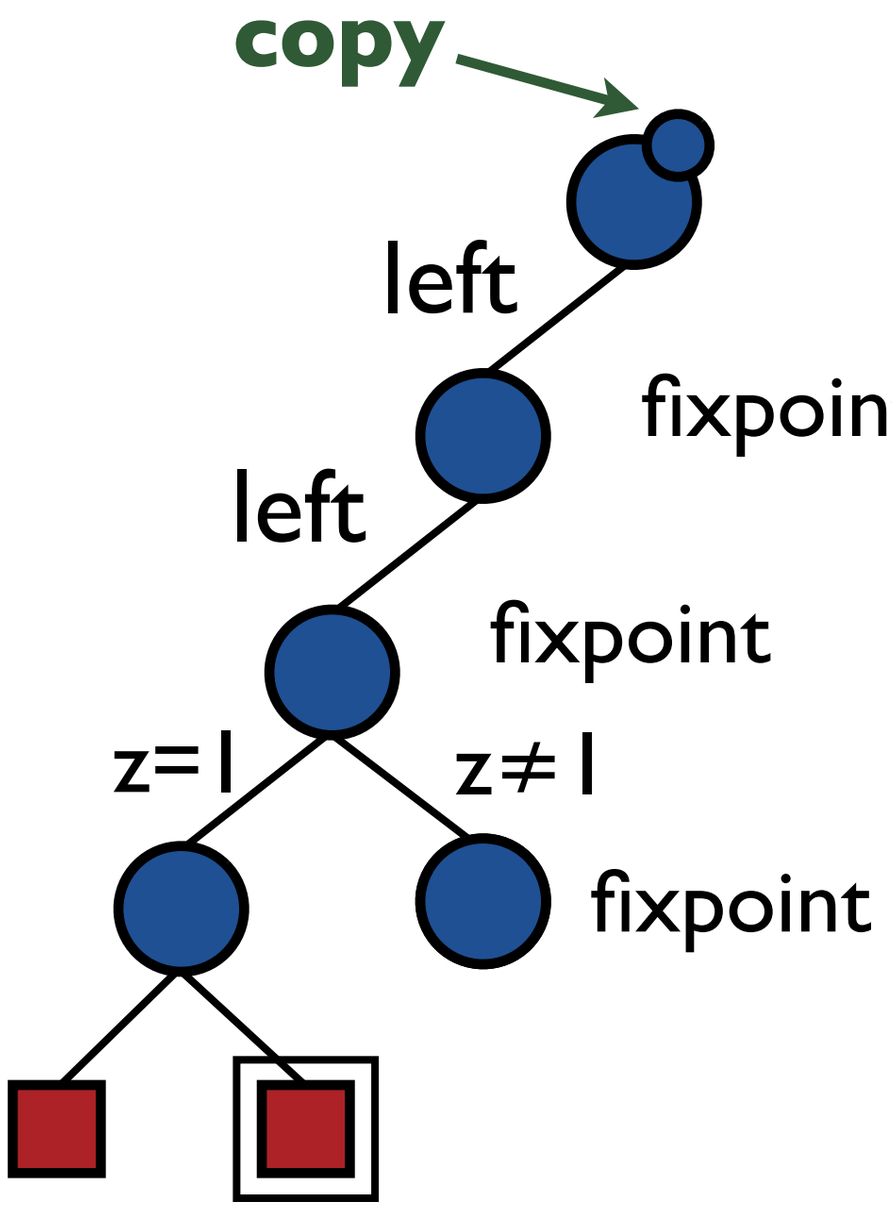

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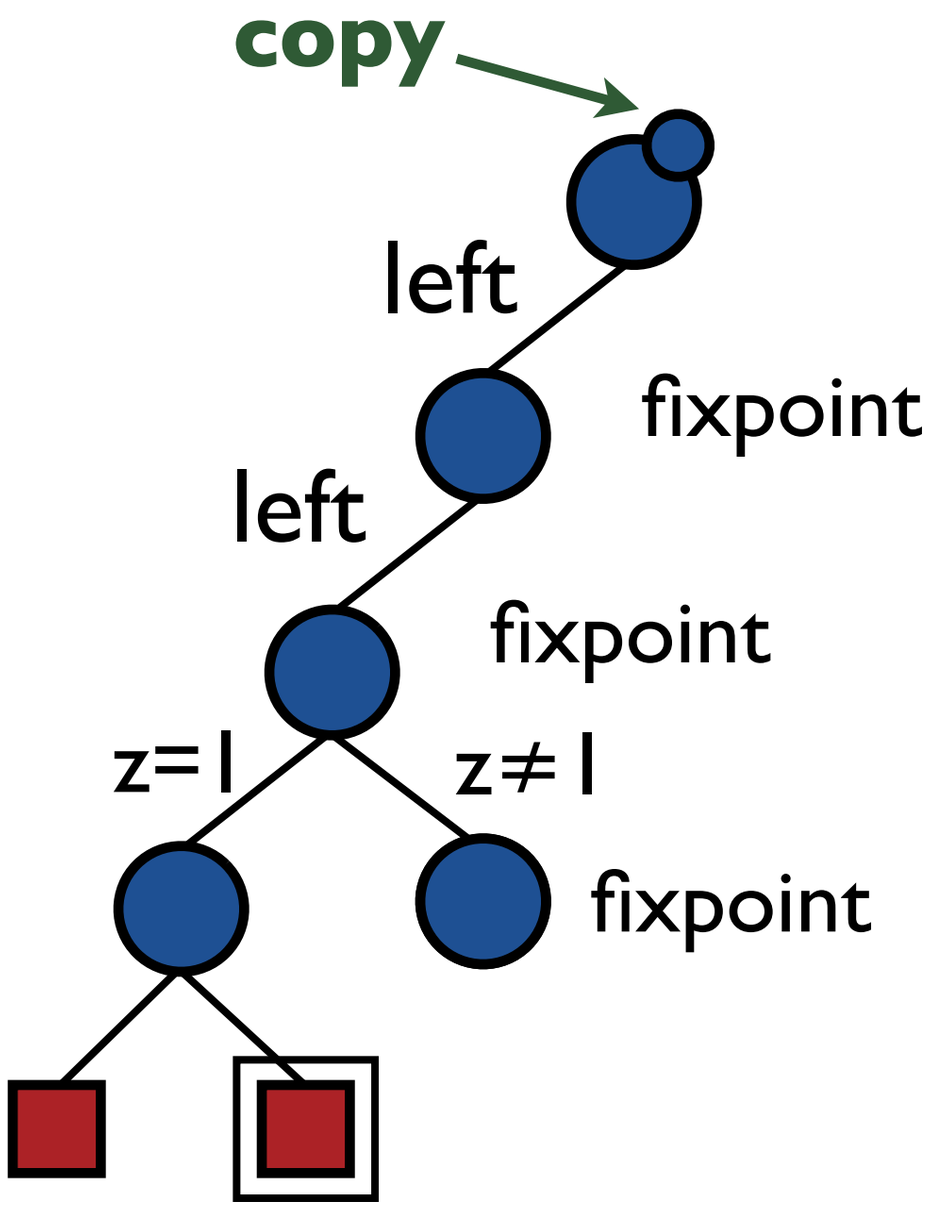

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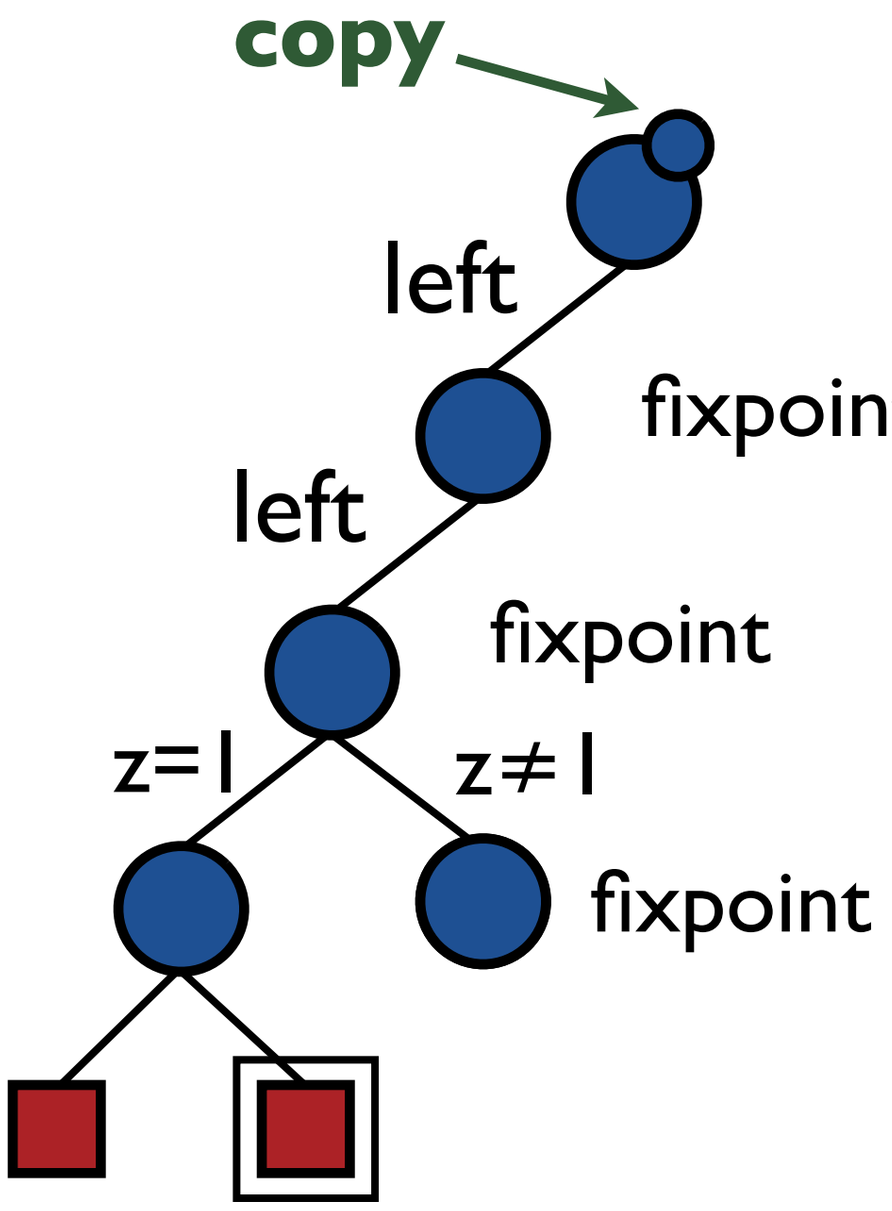

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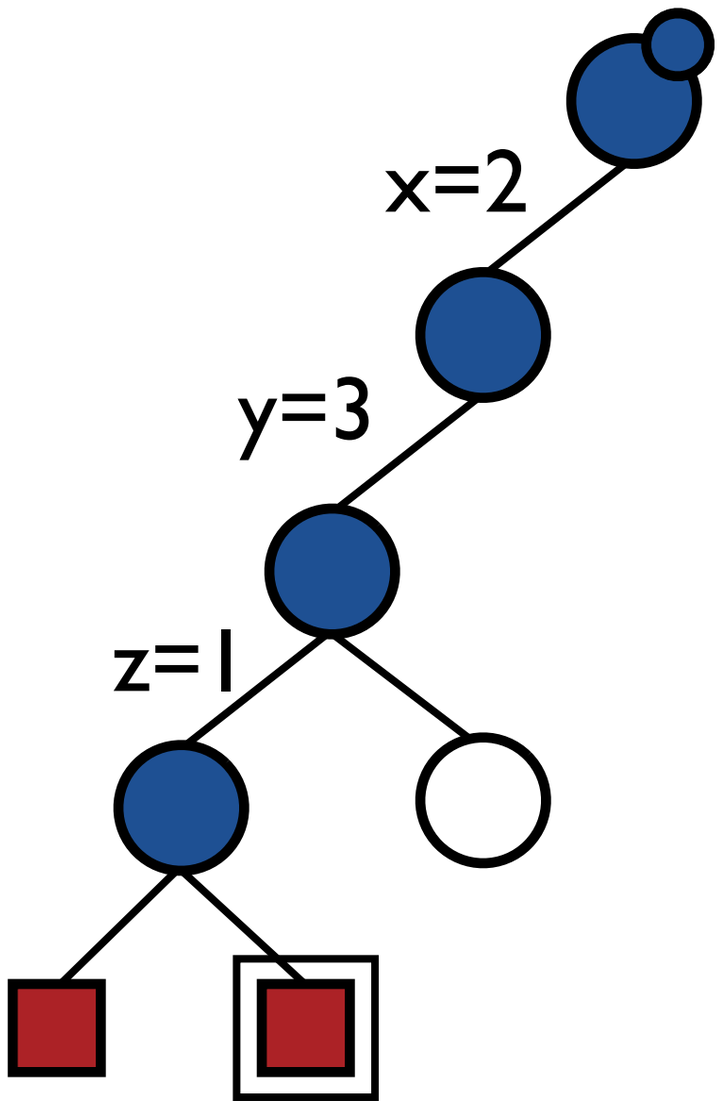

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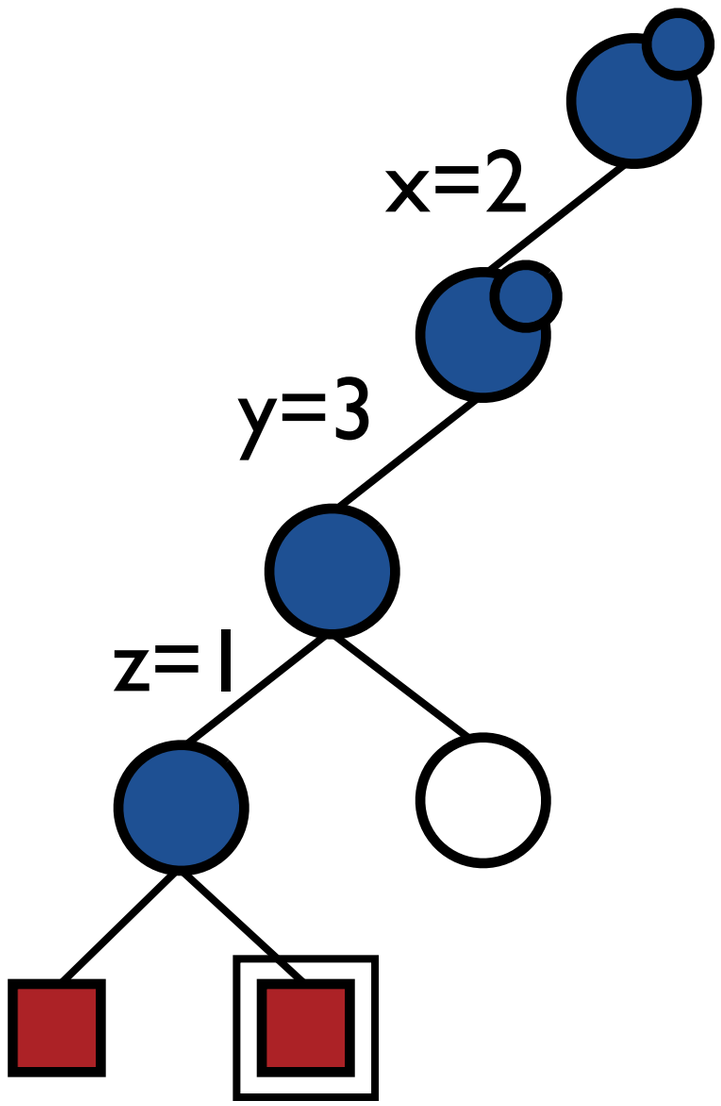

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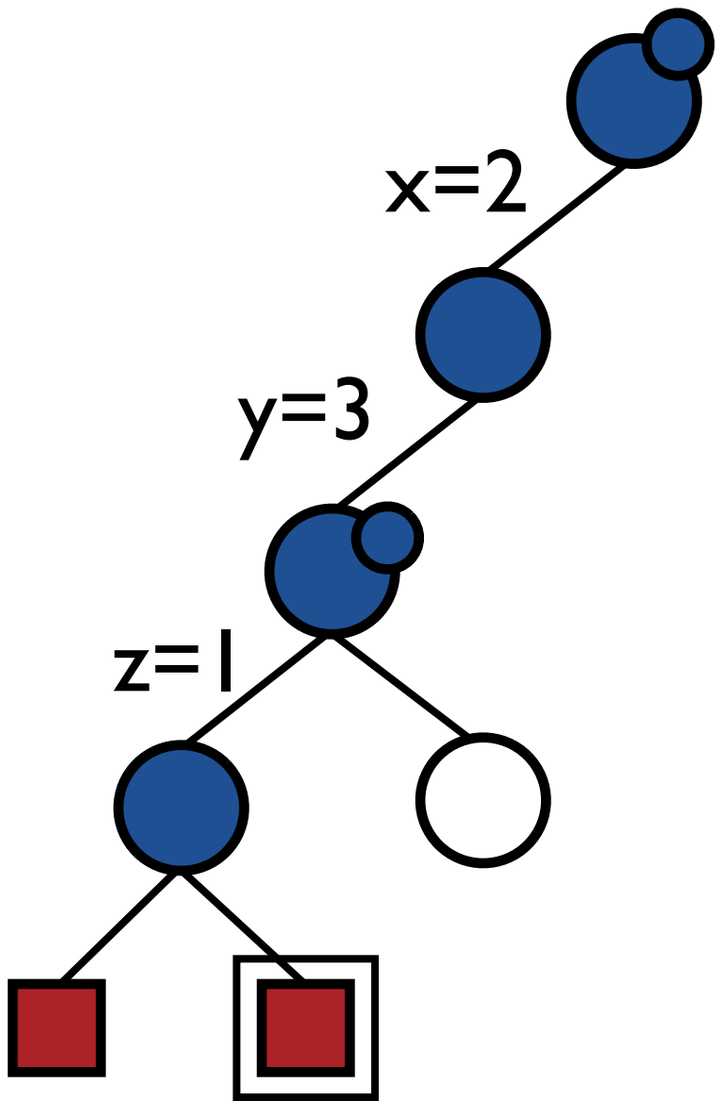

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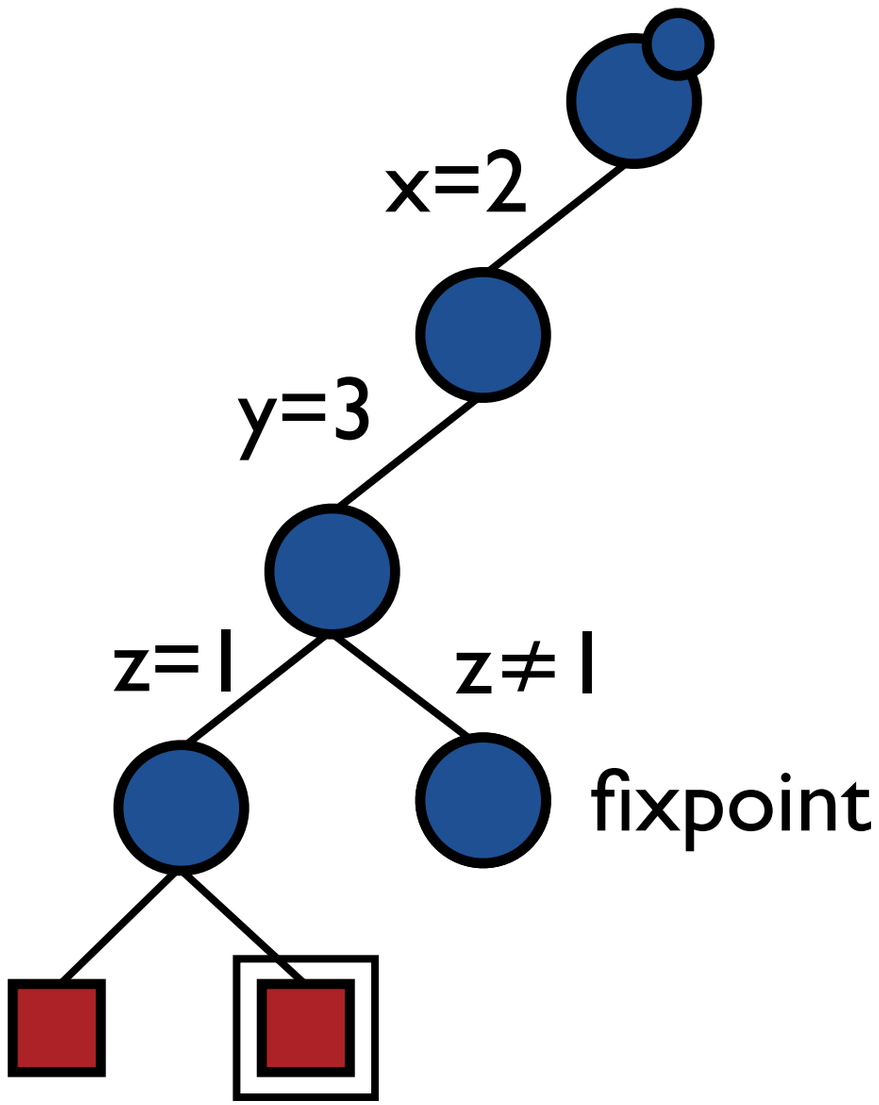

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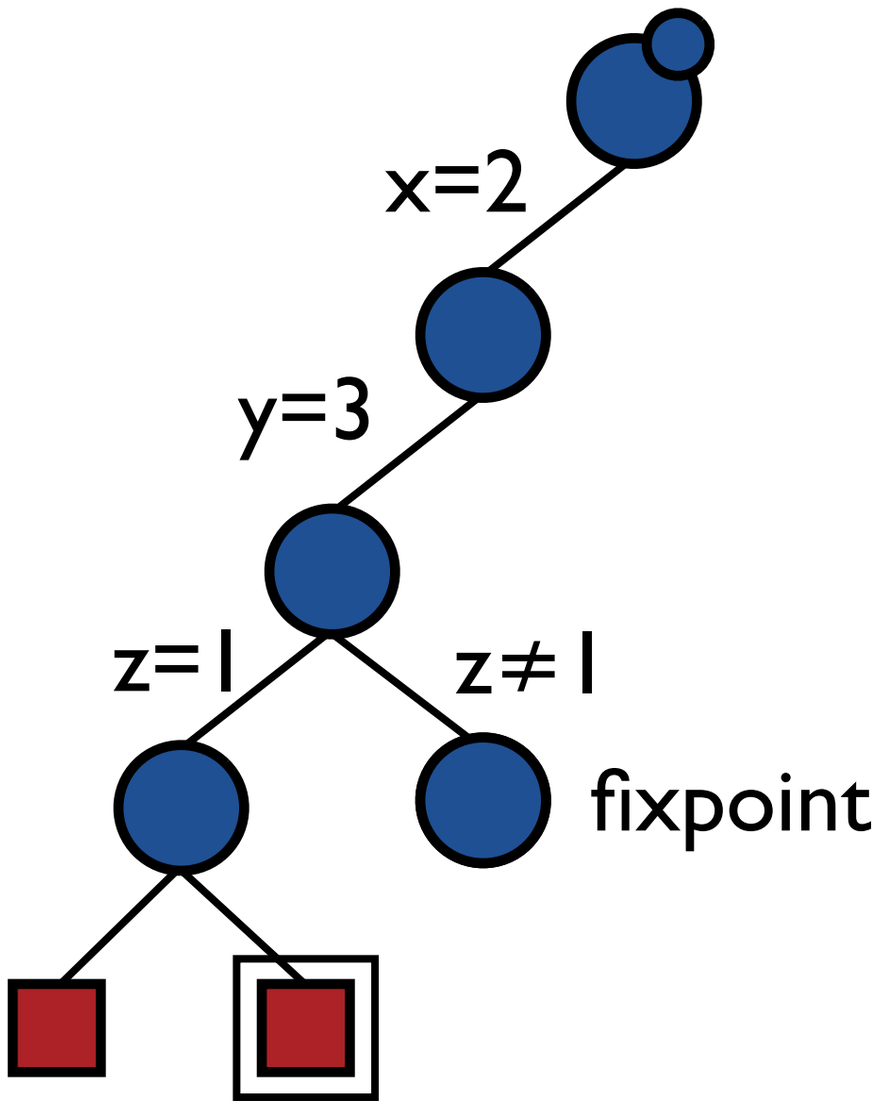

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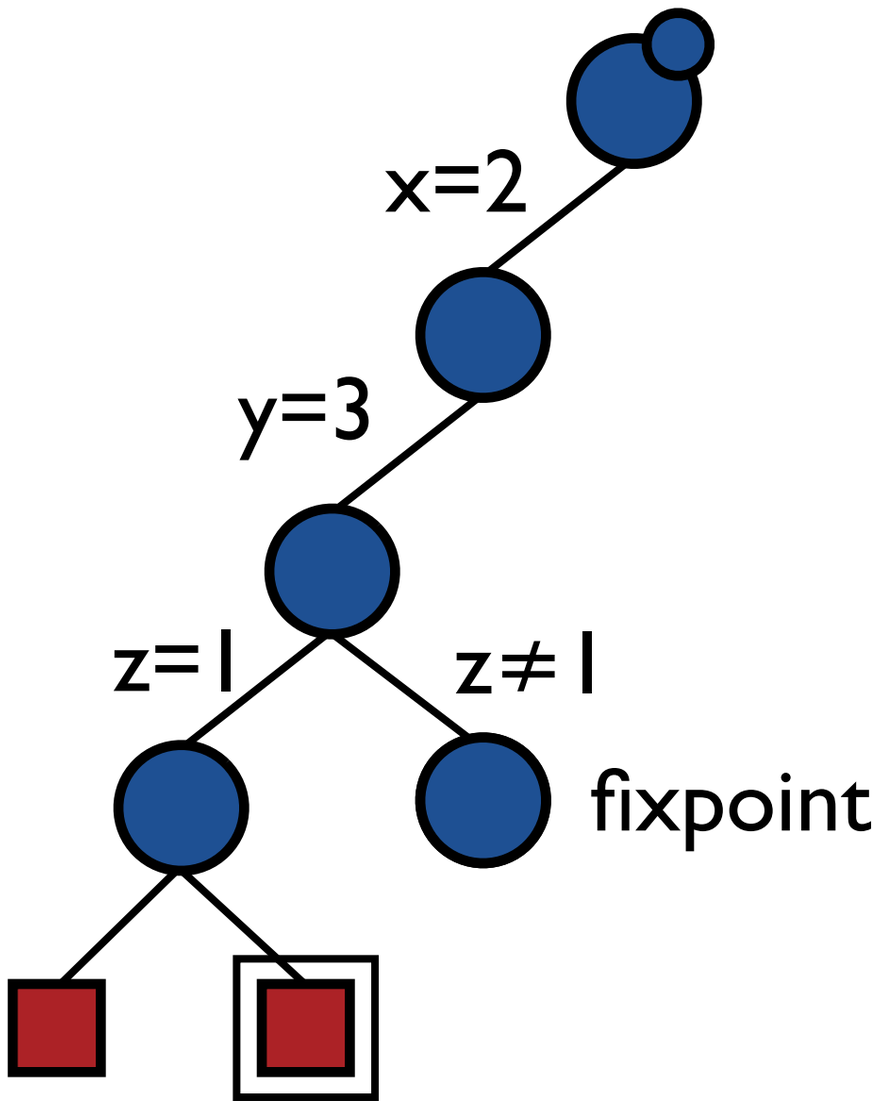

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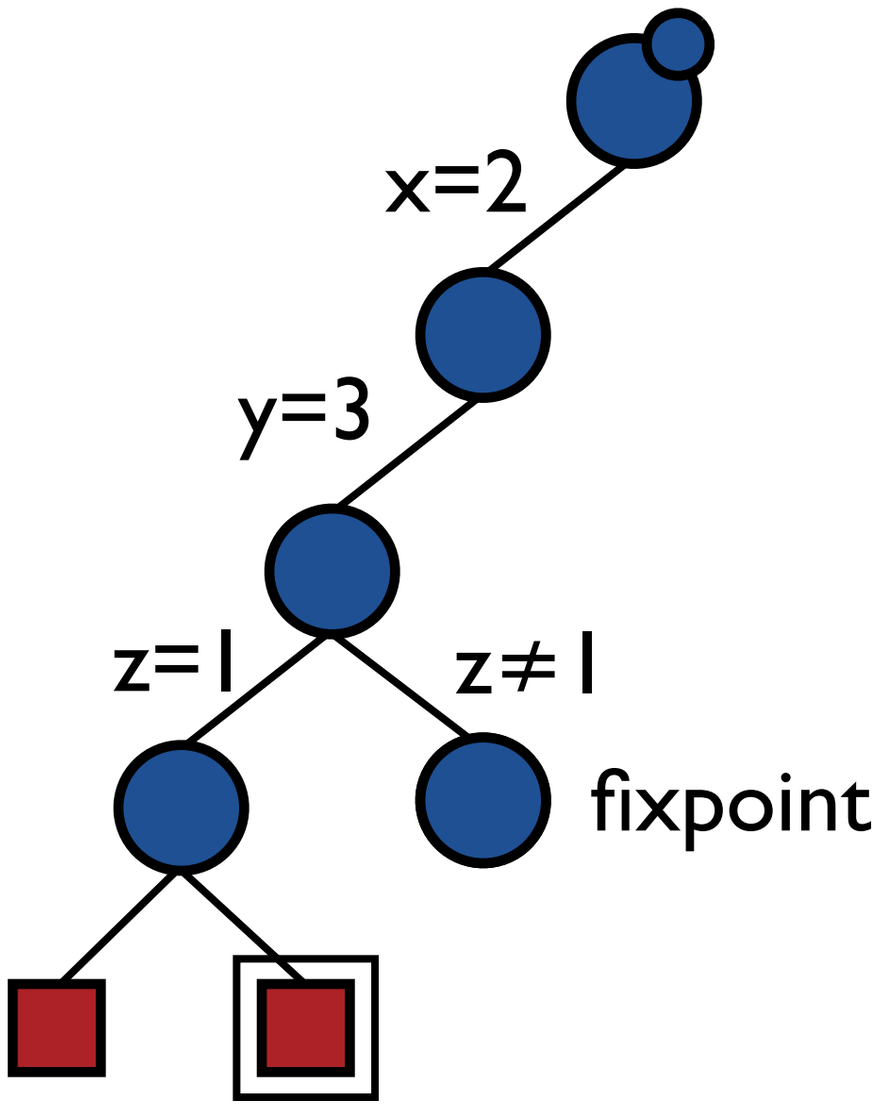

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- **More later in this session**

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