Boolean Equations

George Boole
Laws of Thought
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Gert Smolka, May 4, 2005

4-3

Constants

$0, 1 : \mathbb{B}$
$\neg : \mathbb{B} \to \mathbb{B}$
$\wedge, \vee : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

Standard Interpretation $\mathcal{J}$

$\mathcal{J}\mathbb{B} = \mathbb{B} = \{0,1\}$
$\mathcal{J}0 = 0$
$\mathcal{J}1 = 1$
$\mathcal{J}\neg x = 1 - x$
$\mathcal{J}\wedge x y = \min\{x,y\}$
$\mathcal{J}\vee x y = \max\{x,y\}$

$\boxed{\mathcal{J} \models s = t \quad \text{decidable}}$

since all types denote finite sets

First-order Fragment

BV Boolean variables
set of all variables of type $\mathbb{B}$

BT Boolean terms
set of all terms t such that
- t has type $\mathbb{B}$
- t doesn't contain λ's
- all variables occurring in t are Boolean

BE Boolean equations
set of all equations between $\mathbb{B}$-terms

Modelling with Boolean Equations: Secrets of a Long Live

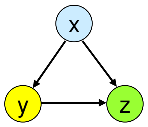
1) If I don't drink beer, I always eat fish
2) If I have both beer and fish, I don't have ice cream
3) If I have ice cream or do not drink beer, I don't have fish

$\neg B \to F = 1$
$B \wedge F \to \neg I = 1$
$I \vee \neg B \to \neg F = 1$

$B = 1$
$\neg F \vee \neg I = 1$

solved form

Modelling with Boolean Equations: Graph Coloring



Colorings of the graph are the solutions of the equations

Is graph bipartite?

$x \neq y$, $x \neq z$, $y \neq z$

$x = \neg y$, …

Is graph 4-partite?

$(x_1, x_2) \neq (y_1, y_2)$, …

$\neg(x_1 \leftrightarrow y_1) \vee \neg(x_2 \leftrightarrow y_2) = 1$, …

Our Main Interest:

Algorithms for solving Boolean equation systems

Boolean Functions

Often, terms and equation systems are used to describe Boolean functions, (e.g., hardware design)

$\sigma \in \Sigma \overset{df}{=} BV \to \mathbb{B}$ Boolean assignments

$f \in BF \overset{df}{=} \Sigma \to \mathbb{B}$ Boolean functions

$\mathfrak{J}t \overset{df}{=} \lambda\sigma.\ \mathfrak{J}\sigma t$ Boolean function described by t

$\mathfrak{J}E \overset{df}{=} \lambda\sigma.\ \sigma \in Sol_{\mathfrak{J}} E$ Boolean function described by E

$\boxed{\mathfrak{J} \models s = t \iff \mathfrak{J}s = \mathfrak{J}t}$

$\boxed{Sol_{\mathfrak{J}} E_1 = Sol_{\mathfrak{J}} E_2 \iff \mathfrak{J}E_1 = \mathfrak{J}E_2}$

Boolean terms and equation systems

- are nice for specifying B. functions
- are not a good data structure for B. functions; for instance, it is difficult to decide whether 2 terms represent the same B. function.

Tautologies and Equivalence

- A Boolean equation $s = t$ is called a **tautology** if $\mathcal{J} \models s = t$
- Two Boolean terms s, t are called **equivalent** if $\mathcal{J} \models s = t$

Notation for Boolean Terms

$$\bar{s} \rightsquigarrow \neg s$$
$$st \rightsquigarrow s \wedge t$$
$$s + t \rightsquigarrow s \vee t$$
$$s \to t \rightsquigarrow \neg s \vee t$$
$$s \leftrightarrow t \rightsquigarrow (s \wedge t) \vee (\neg s \wedge \neg t)$$

Boolean Axioms (BA)

Commutativity	$xy = yx$	$x + y = y + x$
Associativity	$(xy)z = x(yz)$	$(x+y)+z = x+(y+z)$
Distributivity	$x(y+z) = xy + xz$	$x + yz = (x+y)(x+z)$
Identity	$1x = x$	$0 + x = x$
Complement	$x\bar{x} = 0$	$x + \bar{x} = 1$

$$\boxed{\mathcal{J} \models BA}$$

Remark: Associativity can be deduced from the rest.

We know:

$$BA \vdash s = t \implies \mathcal{J} \models s = t$$

We will show:

$$\mathcal{J} \models s = t \implies BA \vdash s = t$$

Deductive completeness of BA for Boolean equations

Deductive Tautologies and Deductive Equivalence

- A Boolean equation $s = t$ is called a **deductive tautology** if $BA \vdash s = t$
- Two Boolean terms s, t are called **deductively equivalent** if $BA \vdash s = t$

$\boxed{s = t \text{ ded tautology} \Rightarrow s = t \text{ tautology}}$

$\boxed{s, t \text{ ded equivalent} \Rightarrow s, t \text{ equivalent}}$

Duality

Dual term $\hat{t}$ is obtained from t by swapping 0 with 1 and $\wedge$ with $\vee$

- $\hat{\hat{t}} = t$
- $\widehat{BA} = BA$
- $E \vdash e \iff \hat{E} \vdash \hat{e}$
- $BA \vdash e \iff BA \vdash \hat{e}$
- $E_1 \overset{BA}{\vdash_0} E_2 \iff \hat{E}_1 \overset{BA}{\vdash_0} \hat{E}_2$

Boolean Algebras

Interpretation satisfying BA (i.e. $\mathfrak{I} \models BA$)

$\mathfrak{I}$ is called **2-valued Boolean algebra**

$\hat{\mathfrak{I}}$: dual Boolean algebra

- $\hat{\hat{\mathfrak{I}}} = \mathfrak{I}$
- $\mathfrak{I} \models e \iff \hat{\mathfrak{I}} \models \hat{e}$
- $BA \models e \iff BA \models \hat{e}$

Power Set Algebras

The power set algebra for a set X is the interpretation defined as follows:

- $\mathcal{D}B = \mathcal{P}X$
- $\mathcal{D}0 = \emptyset$, $\mathcal{D}1 = X$
- $\mathcal{D}\neg A = X - A$
- $\mathcal{D}(\wedge)AB = A \cap B$, $\mathcal{D}(\vee)AB = A \cup B$

$\boxed{\text{Every power set algebra is a Boolean algebra}}$

Stone's Representation Theorem (1936)

$\boxed{\text{Every Boolean algebra is isomorphic to a subalgebra of a power set algebra}}$

$\boxed{\text{Every finite Boolean algebra is isomorphic to a powerset algebra}}$

Proofs will not be given

Some Deductive Tautologies

Idempotence	$xx = x$	$x + x = x$
Dominance	$0x = 0$	$1 + x = 1$
Absorption	$x(x+y) = x$	$x + xy = x$
Negation	$\bar{0} = 1$	$\bar{1} = 0$
de Morgan	$\overline{xy} = \bar{x} + \bar{y}$	$\overline{x+y} = \bar{x}\,\bar{y}$
Double Negation	$\bar{\bar{x}} = x$	
Resolution	$xy + \bar{x}z = xy + \bar{x}z + yz$	$\dots$

BL = BA + the above tautologies

$\boxed{BA \vdash BL}$ $\boxed{DC\ BA = DC\ BL}$

0-1 Theorem

$\{0,1\}$ closed under $\neg, \wedge, \vee$

$$\boxed{\begin{array}{l}\forall t \in BT \text{ that doesn't contain variables:} \\ BA \vdash t = 1 \;\vee\; BA \vdash t = 0\end{array}}$$

Proof: By structural induction on t using the deductive tautologies

$\bar{0} = 1 \qquad \bar{1} = 0$

$0x = 0 \qquad x0 = 0 \qquad 1x = x \qquad x1 = x$

$0 + x = x \qquad x + 0 = x \qquad 1 + x = 1 \qquad x + 1 = 1$ $\square$

More Ded Tautologies

Contraposition $\quad x \to y = \bar{y} \to \bar{x}$

Contradiction $\quad x = \bar{x} \to 0$

Schönfinkel $\quad x \wedge y \to z = x \to (y \to z)$

Modus Ponens

$$x=1,\ x\to y=1 \ \vdash^{BA}_{0}\ y=1$$

Some Conservative Equivalences

$$\boxed{x=y \ \vdash\!\dashv^{BA}_{0}\ x\leftrightarrow y=1}$$

$$\boxed{x=1,\ y=1 \ \vdash\!\dashv^{BA}_{0}\ x\wedge y=1}$$

$$\boxed{x=0,\ y=0 \ \vdash\!\dashv^{BA}_{0}\ x+y=0}$$

$$\boxed{x\to y=1 \ \vdash\!\dashv^{BA}_{0}\ x=xy \ \vdash\!\dashv^{BA}_{0}\ y=y+x}$$ Golden Rule

$$\boxed{xy=0,\ x+y=1 \ \vdash\!\dashv^{BA}_{0}\ x=\overline{y}}$$ Uniqueness of Complements

Normalisation of B. Equation Systems

$$\boxed{\text{For every finite B. eq.-system } E \text{ one can construct a B. term } t \text{ such that } E \vdash\!\dashv^{BA}_{0} t=1}$$

$$\boxed{\text{To solve B. eq. systems, it suffices to simplify B. terms}}$$

Example 1

Claim: $x=0, y=0 \ \vdash\!\dashv^{BA}_{0}\ x+y=0$

Proof: It suffices to prove the following:

1) $x=0, y=0 \ \vdash^{BA}_{0}\ x+y=0$
2) $x+y=0 \ \vdash^{BA}_{0}\ x=0$
3) $x+y=0 \ \vdash^{BA}_{0}\ y=0$

Proof of (3): $x+y=0 \vdash_0^{BA} y=0$

$$
\begin{aligned}
y &= y+0 && \text{Id} \\
&= y+(x+y) && x+y=0 \\
&= x+(y+y) && \\
&= x+y && \text{Idemp} \\
&= 0 && x+y=0
\end{aligned}
$$

BA

Assumption

Example 2

Claim: $BA \vdash 1 = \overline{0}$

Proof: Known: $xy=0,\ x+y=1 \vdash_0^{BA} x=\overline{y}$

Instantiate with $[x:=1,\ y:=0]$

Suffices to prove:

$BA \vdash 1\cdot 0=0,\ 1+0=1$

Both equations are instances of Dominance Laws. $\square$