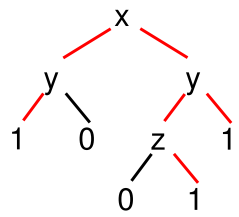
Clause Forms

(DNF, CNF)

Willard V. Quine.
On Cores and Prime Implicants of Truth Functions.
American Mathematical Monthly, 1959.

G. Smolka 6-1 May 20, 2005

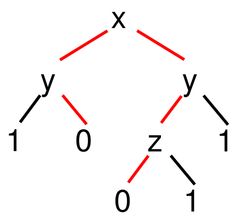
Decision Tree → DNF



$(\neg x \wedge \neg y) \vee (x \wedge \neg y \wedge z) \vee (x \wedge y)$

yields 1 iff one of the clauses yields 1

Decision Tree → CNF



$(x \vee \neg y) \wedge (\neg x \vee y \vee z)$

yields 0 iff one of the clauses yields 0

Conjunctive Normal Forms

$CNF = 1 \mid DC_1 \wedge \cdots \wedge DC_n$ where $n \geq 1$

$DC = 0 \mid L_1 \vee \cdots \vee L_n$ disjunctive clause
where $n \geq 1$ and no variable occurs more than once

$L = x \mid \neg x$ literal

$\boxed{\forall \text{ B. term } \exists \text{ equiv. CNF}}$

Disjunctive Normal Forms

$DNF = 1 \mid CC_1 \vee \dots \vee CC_n$ where $n \geq 1$

$CC = 0 \mid L_1 \wedge \dots \wedge L_n$ **conjunctive clause**
where $n \geq 1$ and no variable occurs more than once

$L = x \mid \neg x$ **literal**

$\boxed{\forall \text{ B. term } \exists \text{ equiv. DNF}}$

$\boxed{\wedge \text{ CNF equiv. to } t \iff \hat{\wedge} \text{ DNF equiv. to } \hat{t}}$

Clause Sets

Good representations for CNFs and DNFs

Clause C: finite set of B. terms

Clause set S: finite set of clauses

Conjunctive interpretation $\bigwedge_{C \in S} \bigvee_{t \in C} t$

Disjunctive interpretation $\bigvee_{C \in S} \bigwedge_{t \in C} t$

$\bigwedge_{\in \emptyset} = 1$, $\bigvee_{\in \emptyset} = 0$

Literal Clause Sets

- clause is called **literal** if it contains only literals
- clause set is called **literal** if it contains only literal clauses

$\boxed{\begin{array}{l} S, S' \text{ literal clause sets:} \\ \underbrace{S, S' \text{ conj. equiv.}} \iff S, S' \text{ disj. equiv} \end{array}}$

Proof: Duality.

equivalent wrt
conjunctive interpretation
(i.e., S, S' describe same B. function)

Normal Clause Sets

- **C trivial** if $\exists t \in C.\ \overline{t} \in C$
- **C normal** if C literal and not trivial
- **S normal** if all clauses of S are normal

$\boxed{\text{Normal clause sets represent CNFs and DNFs}}$

Explicitness

$\boxed{\begin{array}{l} \text{For every normal clause set } S: \\ S \text{ conj. equiv. to } 1 \iff S \text{ disj. equiv. to } 0 \iff S = \emptyset \end{array}}$

CNFs and DNFs are not canonical

$$X = XY + X\bar{Y}$$

$$X = (X+Y)(X+\bar{Y})$$

Will define conjunctive and disjunctive **prime forms** that are canonical
(set representation required)

Redundant Clauses

C **redundant for S** if
either C trivial
or $\exists D \in S.\ D \subseteq C$

Deletion of redundant clauses is equivalence transformation wrt both interpretations

$X + \bar{Y}YZ = X$ (Compl, Id)

$X + Y + YZ = X + Y$ (Absorption)

Resolvents

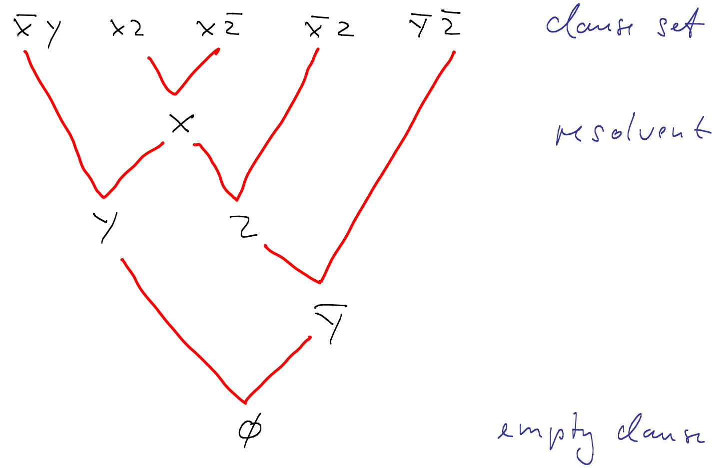
$(C - \{t\}) \cup (D - \{\bar{t}\})$ **resolvent for S**

if $C, D \in S$ and $t \in C$ and $\bar{t} \in D$

Addition of resolvents is equivalence transformation wrt both interpretations

$XY + \bar{X}Z = XY + \bar{X}Z + YZ$ (Resolution)

Example



1) Addition of resolvents and deletion of redundant clauses are equivalence transformations wrt both interpretations

2) Adding non-redundant resolvents and deleting redundant clauses terminates

Proof of Termination

Cla = set of all clauses containing only terms appearing in initial clause set — finite!

Red S = $\{ C \in Cla \mid C$ redundant for $S \}$

- Idea: $Red\ S_1 \subseteq Red\ S_2 \subseteq Red\ S_3 \subseteq \cdots \subseteq Cla$
- Deletion of a redundant clause doesn't change Red S
- Addition of a non-redundant resolvent makes Red S larger.

□

Prime Forms

A clause set is called **prime form** if it

- is literal
- contains no redundant clauses
- has no non-redundant resolvent

Resolution Theorem

$\forall$ prime form S $\forall$ literal clause C:
S equiv. to $S \cup \{C\}$ $\Leftrightarrow$ C redundant for S

Semantic redundancy
= syntactic redundancy

Proof in lecture notes 2004

Consequences of Resolution Theorem

Different prime forms denote different B. functions (wrt both interpretations) — **Canonicity**

For every prime form S:
S conj. equiv. to 0 $\Leftrightarrow$ S disj. equiv. to 1 $\Leftrightarrow$ $S = \{\emptyset\}$ — **Explicitness**

$\forall$ prime form S $\forall$ literal clause set S':
S, S' equiv. $\Rightarrow$ $\forall C \in S'$: C redundant for S

CPFs and DPFs

A prime form s is called a **CPF [DPF] for a term** t if s is equivalent to t wrt the conjunctive [disjunctive] interpretation.

- Every term has exactly one CPF and DPF
- For all B. terms s,t the following are equivalent
  - (1) s and t are equivalent
  - (2) s and t have the same CPF
  - (3) s and t have the same DPF

Proof easy with the previous theorems

Example

$E \rightsquigarrow \{t=1\}$ normalisation
$\rightsquigarrow$ CNF for t
$\rightsquigarrow$ CPF for t
$\rightsquigarrow$ solved form for E

$\bar x \to y = 1$
$xy \to \bar z = 1$
$(z+\bar x) \to \bar y = 1$

x : beer
y : fish
z : ice cream

CNF

$\bar x \to y = x+y$
$xy \to \bar z = \overline{xy} + \bar z = \bar x + \bar y + \bar z$
$(z+\bar x) \to \bar y = \overline{z+\bar x} + \bar y = \bar z x + \bar y$
$= (\bar z + \bar y)(x+\bar y)$

CPF

$xy \quad \bar x \bar y \bar z \quad \bar y \bar z \quad x\bar y$

x

solved form

$x(\bar y + \bar z) = 1$