Solving Boolean Equations with BDDs and Clause Forms

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Abstract

- Methods for solving Boolean equations
  - BDDs [Bryant 1986]
  - Clause forms [Quine 1959]
- Efficient data structure and algorithms for large finite sets (e.g. 2^{1000})

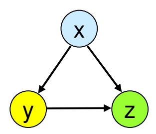
Applications

- Verification (e.g. model checking)
- CAD of HW (e.g. circuit minimization)
- Knowledge representation (e.g. truth maintainance)

Why do I talk about it?

- Beautiful and important
- Interesting trip from logic to algorithms
- Equation solving not covered in textbook accounts of propositional logic
- Had to work it out for our introductory course on Computational Logic

Modelling with Boolean Equations: Graph Coloring



Colorings of the graph are the solutions of the equations

Is graph bipartite?

$x \neq y$, $x \neq z$, $y \neq z$

$x = \neg y$, …

Is graph 4-partite?

$(x_1,x_2) \neq (y_1,y_2)$, …

$\neg(x_1 \leftrightarrow y_1) \vee \neg(x_2 \leftrightarrow y_2) = 1$, …

Modelling with Boolean Equations: Secrets of a Long Live

1) If I don't drink beer, I always eat fish
2) If I have both beer and fish, I don't have ice cream
3) If I have ice cream or do not drink beer, I don't have fish

$\neg B \rightarrow F = 1$

$B \wedge F \rightarrow \neg I = 1$

$I \vee \neg B \rightarrow \neg F = 1$

$B = 1$

$\neg F \vee \neg I = 1$

solved form

Formalities

Bool = {0,1}
x,y,z ∈ Var
s ∈ State = Var→Bool
f,g ∈ BF = State→Bool

BF ≅ P(State) $\qquad$ {s∈State | fs=1}

a,b,c ∈ Exp
Den ∈ Exp→BF

Boolean Operations

$Bool^n \rightarrow Bool$

$x \wedge y = \min\{x,y\}$
$x \vee y = \max\{x,y\}$
$\neg x = 1-x$
$x \rightarrow y$ = if $x \leq y$ then 1 else 0
$x \leftrightarrow y$ = if $x=y$ then 1 else 0

Solving Equation Systems

$$\begin{tikzcd} \text{ESys} \arrow[d] & \text{Exp} \arrow[d] \\ \text{P(State)} \arrow[r, phantom, "\cong"] & \text{BF} \end{tikzcd}$$

Solutions of equation system can be described by Boolean function

Solving Equation Systems (2)

$$\begin{tikzcd} \text{ESys} \arrow[r] \arrow[d] & \text{Exp} \arrow[d] \\ \text{P(State)} \arrow[r, phantom, "\cong"] & \text{BF} \end{tikzcd}$$

Phase 1: equation system $\rightarrow$ expression

Solving Equation Systems (3)

$$\begin{tikzcd} \text{ESys} \arrow[r] \arrow[d] & \text{Exp} \arrow[d] \arrow[dr] & \\ \text{P(State)} \arrow[r, phantom, "\cong"] & \text{BF} \arrow[r, phantom, "\cong"] & \text{Rep} \end{tikzcd}$$

Phase 2: expression $\rightarrow$ good rep of BF

Solving Equation Systems (4)

$$\begin{tikzcd} \text{ESys} \arrow[r] \arrow[d] & \text{Exp} \arrow[d] \arrow[r, phantom, "\subseteq"] & \text{Exp'} \arrow[d, phantom, "\cup\!\mid"] \\ \text{P(State)} \arrow[r, phantom, "\cong"] & \text{BF} \arrow[r, phantom, "\cong"] & \text{Rep} \end{tikzcd}$$

Extend expressions to contain good reps of BFs

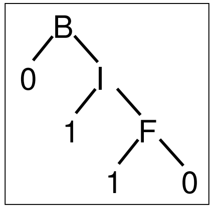
Equation System → Expression

$a=b$	$\Leftrightarrow$	$a\leftrightarrow b=1$
$a\neq b$	$\Leftrightarrow$	$\neg a\leftrightarrow b=1$
$a\leq b$	$\Leftrightarrow$	$a\rightarrow b=1$
$a<b$	$\Leftrightarrow$	$\neg a\wedge b=1$
$a=1$ and $b=1$	$\Leftrightarrow$	$a\wedge b=1$
$a=1$ or $b=1$	$\Leftrightarrow$	$a\vee b=1$

Example

$$\begin{aligned} \neg B\rightarrow F &= 1 \\ B\wedge F \rightarrow \neg I &= 1 \\ I\vee\neg B \rightarrow \neg F &= 1 \end{aligned}$$

Equation system



Prime tree

$$(\neg B\rightarrow F) \wedge (B\wedge F\rightarrow\neg I) \wedge (I\vee\neg B \rightarrow \neg F) = 1$$

Normal equation

$B \wedge (\neg F\vee\neg I)$

Conjunctive prime form

$(B\wedge\neg I) \vee (B\wedge\neg F)$

Disjunctive prime form

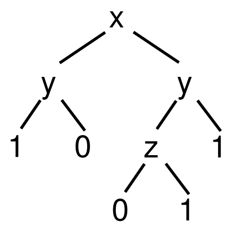
Overview

- Intro
- BDDs [Bryant 1986]
- Clause forms

BDDs

- Decision trees
- Prime trees
- Algorithms
- Minimal Graph Representation

Decision Trees



Graphical Representation
of Nested Conditionals

```
if x=0
then if y=0
      then 1
      else 0
else if y=0
      then if z=0
            then 0
            else 1
      else 1
```

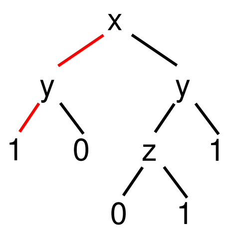
Conditonal as new Operation

$Bool^3 \rightarrow Bool$

$$
\begin{aligned}
(x,y,z) &= \text{if } x{=}0 \text{ then } y \text{ else } z \\
&= (\neg x \rightarrow y) \wedge (x \rightarrow z) \\
&= (\neg x \wedge \neg y) \vee (x \wedge z)
\end{aligned}
$$

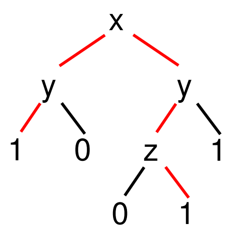
[Löwenheim 1910]

Decision Tree → DNF



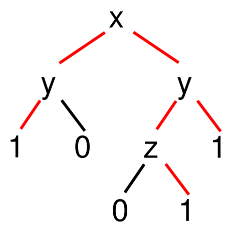
$(\neg x \wedge \neg y) \vee \ldots$

Decision Tree → DNF



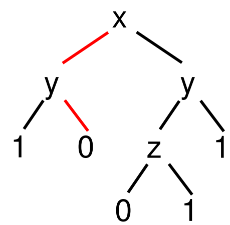
$(\neg x \wedge \neg y) \vee (x \wedge \neg y \wedge z) \vee \ldots$

Decision Tree → DNF



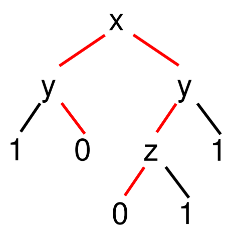
$(\neg x \wedge \neg y) \vee (x \wedge \neg y \wedge z) \vee (x \wedge y)$

Decision Tree → CNF



$(x \vee \neg y) \wedge \ldots$

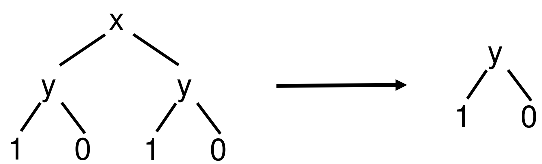
Decision Tree → CNF



$(x \vee \neg y) \wedge (\neg x \vee y \vee z)$

Reduction of Decision Trees

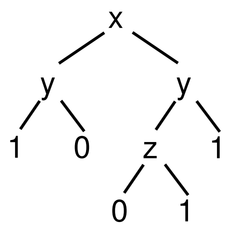
Based on $(x,y,y) = y$



Ordered Decision Trees

- Fix linear order on variables

  $x < y < z < \ldots$
- Deeper variables must be larger



Prime Trees

- Ordered and reduced decision trees
- Isomorphic to Boolean functions
- Perfect representation of Boolean functions

$$\begin{tikzcd} \mathrm{Exp} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathrm{Exp'} \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\ \mathrm{BF} \arrow[r, phantom, "\cong"] & \mathrm{PT} \end{tikzcd}$$

Theorem Different prime trees denote different Boolean functions.

Proof By induction on max of sizes. Case analysis:

1. a and b are both atomic.
2. Root variables of a and b are identical.
3. Root variable of a does not occur in b.

Theorem Every expression can be translated into equivalent prime tree.

Expansion Theorem

(Boole 1854, Löwenheim 1910, Shannon 1938)

$$a \equiv (x, a[x{:=}0], a[x{:=}1])$$

Operations on Prime Trees

not: PT $\rightarrow$ PT
not a = $\pi(\neg a)$

and: PT$\times$PT $\rightarrow$ PT
and(a,b) = $\pi(a \wedge b)$

Will see efficient algorithms

Constructors for PTs (ADT)

0: PT
1: PT
cond: Var$\times$PT$\times$PT $\rightarrow$ PT
cond(x,a,b) = π(x,a,b) provided x<Va$\cup$Vb

If a,b prime trees and x variable:

π(x,a,a) = a
π(x,a,b) = (x,a,b) if x<Va$\cup$Vb

All algorithms will be based on these constructors

Algorithm for not

- Based on

  $\neg 0 = 1$

  $\neg 1 = 0$

  $\neg(x,y,z) = (x,\neg y,\neg z)$

- Orderedness preserved since no new variables
- Reducedness preserved since not injective

Algorithm for and

- Based on

  $(x,a,b) \wedge 0 = 0$

  $(x,a,b) \wedge 1 = (x,a,b)$

  $(x,a,b) \wedge (x,a',b') = (x, a\wedge a', b\wedge b')$

  $(x,a,b) \wedge c = (x, a\wedge c, b\wedge c)$ (only used if x < Vc)

- Orderedness preserved since no new variables
- Reducedness preserved by cond

Expression → Prime Tree

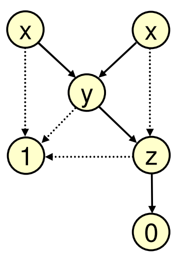
trans: Exp → PT

trans 0 = 1
trans 1 = 1
trans x = cond(x,0,1)
trans (¬a) = not(trans a)
trans (a∧b) = and(trans a, trans b)

As is, and is exponential

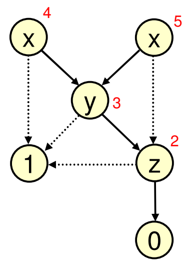
- Can make it quadratic by
  - dynamic programming (hashing over PTs)
  - constant time equality test for PTs

Minimal Graph Representation



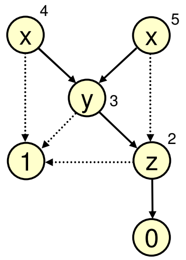
- Every node describes a prime tree
- Graph describes a subtree-closed set of prime trees
- Graph minimal iff different nodes describe different trees

Graph → Table



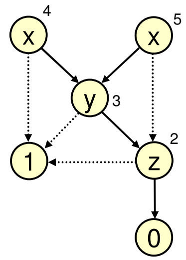
Number nodes of graph

Graph → Table



2	(z,1,0)
3	(y,1,0)
4	(x,1,3)
5	(x,2,3)

Graph → Table → Function



i	tab(i)
2	(z,1,0)
3	(y,1,0)
4	(x,1,3)
5	(x,2,3)

Graph minimal iff tab injective

Constant Time Realization of cond

cond(x,n,n') =
  if n=n' then n
  else if (x,n,n') $\in$ Dom(tab^{-1})
    then tab^{-1} (x,n,n')
    else let n'' = least number not in Dom tab
      in tab := tab[n'':=(x,n,n')] ;
        n''

Implement tab^{-1} with hashing

Overview

- Intro
- BDDs
- Clause forms [Quine 1959]

Conjunctive Normal Forms

literal	x, $\neg x$
clause C	finite set of literals, not x and $\neg x$
clause set S	finite set of clauses
cnf S	new expression form

$$(\text{cnf } S)s \;=\; \bigwedge_{C \in S} \bigvee_{a \in C} as \qquad (\wedge\varnothing = 1,\ \vee\varnothing = 0)$$

Conjunctive Prime Forms

- C implicate of a $\Leftrightarrow$ $a \le \vee C$
- C prime implicate of a $\Leftrightarrow$ C minimal implicate of a
- Formula has only finitely many prime implicates
- $a \equiv \text{cnf}\ \{C \mid C \text{ prime implicate of } a\}$

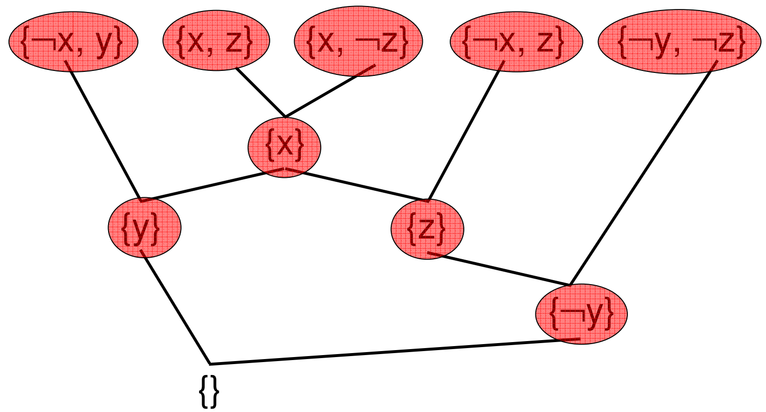
$$\begin{tikzcd} \text{Exp} \arrow[r, phantom, "\subseteq"] \arrow[d] & \text{Exp'} \arrow[d, phantom, "\cup|"] \\ \text{BF} \arrow[r, phantom, "\cong"] & \text{CPF} \end{tikzcd}$$

CNF $\rightarrow$ CPF

- CPF can be computed from CNF by 2 rules:
  - delete subsumed clause
  - add resolvent that is not subsumed

    $(a \vee b) \wedge (\neg a \vee c) \;\le\; (b \vee c)$
- Equivalence transformations
- Terminate with CPF

Example : CNF $\rightarrow$ CPF



CNF $\rightarrow$ CPF

- Nice for few variables
- Explosive in number of variables
- By duality: DNF $\rightarrow$ DPF
- Application: truth maintainance in AI (CPF)
  - Reiter and de Kleer 1987
- Application: circuit minimization (DPF)
  - Quine 1959
  - Minimal size DNFs are subsets of DPF

Summary and Remarks

- 2 Methods for solving Boolean equations
  - BDDs [Bryant 1986]
  - clause forms [Quine 1959]
- Generalizes to Boolean algebras
- Generalizes to infinitely many variables
- There are other methods, e.g.
  - Complete normal forms [Boole 1854]
  - [Löwenheim 1910]

References

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