$$\textbf{Ref}\ \frac{}{s = s} \qquad \textbf{Sym}\ \frac{s = t}{t = s} \qquad \textbf{Trans}\ \frac{s = s' \quad s' = t}{s = t}$$

$$\textbf{CL}\ \frac{s = s'}{st = s't} \qquad \textbf{CR}\ \frac{t = t'}{st = st'} \qquad \xi\ \frac{s = s'}{\lambda x.s = \lambda x.s'}$$

$$\beta\ \frac{}{(\lambda x.s)t = s[x{:=}t]} \qquad \eta\ \frac{}{\lambda x.sx = s}\ x \notin \mathcal{N}s$$

Figure 2: Deduction rules

4 Equational Deduction

Given an equational specification, one can infer semantically entailed equations by "replacing equals with equals", a proof method known as equational deduction. Equational deduction is a syntactic proof method since it is based on syntactic rules rather than semantic arguments.

Figure 2 shows the so-called **deduction rules**. Each deduction rule states a pattern according to which an equation (the **conclusion** below the bar) can be obtained from given equations (the **premises** above the bar). Formally, each rule describes a set of pairs (E, e) (the **instances of the rule**) where E is the set of premises and e is the conclusion. The rules ξ and η, for instance, describe the following sets of instances:

$$\xi: \quad \{\, (\{s = s'\},\ \lambda x.s = \lambda x.s') \mid x \in \mathit{Var}\ \wedge\ s, s' \in \mathit{Ter}\ \wedge\ \tau s = \tau s' \,\}$$
$$\eta: \quad \{\, (\emptyset,\ \lambda x.sx = s) \mid s \in \mathit{Ter}\ \wedge\ x \notin \mathcal{N}s \,\}$$

The rules Ref, Sym and Trans provide the equivalence properties of equality. The rules CL, CR and ξ provide the so-called congruence properties of equality. They make it possible to replace equals with equals within a term. Note that rule ξ exploits the fact that variables are universally quantified (x may occur in s and s'). Rule β and η provide basic equational properties of abstractions we have discussed before. The fundamental property of the deuction rules is soundness:

Proposition 4.1 (Soundness) If (E, e) is an instance of a deduction rule, then $E \models e$.

A **derivation of** e **from** A is a tuple $(e_1, \dots, e_n)$ such that $e = e_n$ and for every $i \in \{1, \dots, n\}$: $e_i \in A$ or there exists a set $E \subseteq \{e_1, \dots, e_{i-1}\}$ such that (E, e_i)

is an instance of a deduction rule. We can now define **deductive entailment** as follows:

$A \vdash e$	$:\Longleftrightarrow$ $\exists$ derivation of e from A	A **entails** e **deductively**
$A \vdash E$	$:\Longleftrightarrow$ $\forall e \in E\colon\ A \vdash e$	A **entails** E **deductively**

Proposition 4.2 (Soundness) $A \vdash e \implies A \vDash e$

Deductive equivalence of specifications is defined as follows:

$A \dashv\vdash A' \;:\Longleftrightarrow\; A \vdash A' \;\wedge\; A' \vdash A$ $\qquad$ A, A' **deductively equivalent**

By the soundness property we know that deductive equivalence implies semantic equivalence:

Proposition 4.3 $A \dashv\vdash A' \implies A \;\rightleftharpoons\; A'$

Proposition 4.4 (Extensionality)

1. $\{\lambda x.s = \lambda x.t\} \dashv\vdash \{s = t\}$
2. $x \notin \mathcal{N}(s = t) \implies \{sx = tx\} \dashv\vdash \{s = t\}$

Proof Here is a derivation that proves $\vdash$ of (1):

$\lambda x.s = \lambda x.t$	
$(\lambda x.s)x = (\lambda x.t)x$	CL
$(\lambda x.s)x = s$	β
$s = (\lambda x.s)x$	Sym
$(\lambda x.t)x = t$	β
$s = (\lambda x.t)x$	Trans
$s = t$	Trans

The other proofs are similar. Exercise! ■

Proposition 4.5 (Finiteness) If $A \vdash e$, then there exists a finite subset $A' \subseteq A$ such that $A' \vdash e$.

Example 4.6 Let $fyx = a$ be an equation where f and a are constants and x and y are variables such that $\tau x = \tau y$. The following outlines a derivation of

$fyx = a$ from $\{fxy = a\}$.

	$fxy = a$	
$\vdash$	$\lambda x.fxy = \lambda x.a$	ξ
$\vdash$	$\lambda yx.fxy = \lambda yx.a$	ξ
$\vdash$	$(\lambda yx.fxy)x = (\lambda yx.a)x$	CL
$\vdash$	$\lambda x'.fx'x = \lambda x.a$	β, Sym, Trans
$\vdash$	$(\lambda x'.fx'x)y = (\lambda x.a)y$	CL
$\vdash$	$fyx = a$	β, Sym, Trans ■

The example suggests that we can deduce from e every instance of e that is obtained by instantiation of some variables of e. This property is called *generativity*. We will make use of the following notation:

$Ker\theta \;:=\; \{\, u \in Ind \mid \theta u \neq u \,\}$ **Kernel of** θ

Proposition 4.7 (Generativity) $Ker\theta \subseteq Var \implies \{e\} \vdash \mathbf{S}\theta e$

This proposition can be proven with the following lemma:

Lemma 4.8 $\emptyset \vdash \mathbf{S}\{x_1{:=}s_1, \dots, x_n{:=}s_n\}t = (\lambda x_1 \dots x_n.\, t)s_1 \dots s_n$

Proof By induction on n. ■

Deductive generativity implies semantic generativity (by soundness):

Proposition 4.9 (Generativity) $Ker\theta \subseteq Var \implies \{e\} \vDash \mathbf{S}\theta e$

A substitution θ is **invertible** if there exists a substitution ψ such that $\mathbf{S}\psi(\mathbf{S}\theta s) = s$ for all terms s. A **variable renaming** is an invertible substitution θ such that $Ker\theta \subseteq Var$.

Proposition 4.10 (Variable Renaming) θ variable renaming $\implies \{e\} \vdash\dashv \{\mathbf{S}\theta e\}$

Proof Easy consequence of Generativity. ■

Another important property of the entailment relations is *stability*. We say that a deduction rule is **stable** if for every instance (E, e) of the rule and every substitution θ the pair $(\mathbf{S}\theta E,\ \mathbf{S}\theta e)$ is an instance of the rule.

Proposition 4.11 All deduction rules but ξ are stable.

We say that a substitution θ is **stable** for an equation e if it satisfies the following conditions:

1. $Ker\theta \subseteq Con$
2. $\forall c \in \mathcal{N}e\ \forall x \in \mathcal{N}(\mathbf{S}\theta c):\ x \notin \mathcal{N}e$

We say that a substitution θ is **stable** for a set of equations E if θ is stable for every equation in E.

Proposition 4.12 If $Ker\theta \subseteq Con$ and θc is closed for all constants c, then θ is stable for every equation.

Proposition 4.13 (Stability) Let θ be stable for A. Then:

1. $A \vdash e \implies \mathbf{S}\theta A \vdash \mathbf{S}\theta e$
2. $A \vDash e \implies \mathbf{S}\theta A \vDash \mathbf{S}\theta e$

The proof of this proposition is not straightforward.

Example 4.14 By Generativity we know $\{fax = x\} \vdash fay = y$. The substitution $\theta = \{a := x\}$ is not stable for $\{fax = x\}$ and in fact $\{fxx = x\} \nvdash fxy = y$ since there is structure $\mathcal{A}$ such that $\mathcal{A} \vDash fxx = x$ and $\mathcal{A} \nvDash fxy = y$. Exercise: Find such a structure. ■

A **duality** for a specification A is a substitution δ such that:

1. δ stable for A
2. $\forall s:\ A \vdash \mathbf{S}\delta(\mathbf{S}\delta s) = s$
3. $A \vdash \mathbf{S}\delta A$

Proposition 4.15 (Duality) Let δ be a duality for A. Then:

1. $A \vdash e \iff A \vdash \mathbf{S}\delta e$
2. $A \vDash e \iff A \vDash \mathbf{S}\delta e$

Proof We proof (1) as follows:

$$\begin{aligned} A \vdash e &\implies \mathbf{S}\delta A \vdash \mathbf{S}\delta e && \text{stability} \\ &\implies A \vdash \mathbf{S}\delta e && \delta \text{ duality, } (3) \\ &\implies \mathbf{S}\delta A \vdash \mathbf{S}\delta(\mathbf{S}\delta e) && \text{stability} \\ &\implies A \vdash e && \delta \text{ duality, } (3) \text{ and } (2) \end{aligned}$$

The proof of (2) is similar and exploits soundness. ■

Example 4.16 $\delta = \{0{:=}1,\ 1{:=}0,\ +{:=}\cdot,\ \cdot{:=}+\}$ is a duality for BA that satisfies $\mathbf{S}\delta(\text{BA}) = \text{BA}$ and $\mathbf{S}\delta(\mathbf{S}\delta s) = s$. ■